

Affine actions on non-archimedean trees

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Abstract

We initiate the study of affine actions of groups on Λ -trees for a general ordered abelian group Λ ; these are actions by dilations rather than isometries. This gives a common generalisation of isometric action on a Λ -tree, and affine action on an \mathbb{R} -tree as studied by I. Lioussé. The duality between based length functions and actions on Λ -trees is generalised to this setting. We are led to consider a new class of groups: those that admit a free affine action on a Λ -tree for some Λ . Examples of such groups are presented, including soluble Baumslag-Solitar groups and the discrete Heisenberg group.

Introduction

This paper attempts to draw together three strands: the study of affine actions on \mathbb{R} -trees, which were first considered by I. Lioussé in [20]; Lyndon length functions (based length functions) on groups and the duality between such length functions and isometric actions on Λ -trees; and a desire to rehabilitate certain examples of groups (studied in [26]) that admit a free non-nesting action on a pretree but which admit no free isometric action on a Λ -tree.

Isometric actions on Λ -trees now form a well-established theme in geometric group theory. This area has its origins in a paper of R. Lyndon [21], where Nielsen's proof of the classical Nielsen-Schreier Theorem was presented in terms of abstract length functions on groups, and the Kurosh Subgroup Theorem was also deduced from this framework. In fact, anticipating future developments in the area, Lyndon commented that “we feel strongly that the restriction to length functions whose values are integers, rather than real numbers, elements of an ordered abelian group, or even of algebraic structures of a more unfamiliar nature is regrettable. . .”

A crucial step in the development of the theory was the development of Bass-Serre theory in the late 1960s. This theory established an equivalence between group actions (without inversions) on trees and graphs of groups (the latter are a common generalisation of amalgamated free products and HNN extensions), and provided new proofs of many results and a new viewpoint enabling the development of others.

The link between group actions on trees and length functions was provided by I.M. Chiswell [7], who showed that specifying a Lyndon length function on a group is essentially equivalent to giving an action of the group on a tree. One now had three strikingly different conceptual approaches to a common phenomenon: length functions, actions on trees, and graphs of groups. This framework had the effect of bringing many geometric and group theoretic themes closer together. For example these ideas gave a common conceptual viewpoint for understanding the two original proofs of the Nielsen-Schreier subgroup theorem.

The theory of \mathbb{R} -trees and their interaction with group theory has become a very important theme over the last three decades. We refer to [4] for more information about this rich branch of the subject. Suffice it to say that Λ -trees, and with them \mathbb{R} -trees, were first defined by J. Morgan and P. Shalen in the course of their work on algebraic subvarieties of $\mathrm{SL}(2, \mathbb{C})$ -characters of finitely generated groups. (It is worth mentioning however that the notion of \mathbb{R} -trees had been implicit beforehand in the work of W. Imrich [15] and J. Tits [30], as well as the work of Chiswell cited above.)

Free actions have always been of special interest. A group that admits a free action (without inversions) on a Λ -tree is said to be Λ -free. This terminology stems from the fact that in the

classical case $\Lambda = \mathbb{Z}$, a group is Λ -free if and only if it is free. The question of which groups are \mathbb{R} -free was the subject of intense investigation culminating in Rips' Theorem: a finitely generated group is \mathbb{R} -free if and only if it is a free product of finitely generated free abelian groups and surface groups (with the exception of the fundamental groups of non-orientable surfaces of genus ≤ 3).

It is of interest to note here that finitely generated fully residually free groups are Λ -free, where Λ can be taken to have the form \mathbb{Z}^n . Moreover \mathbb{R} -free groups are locally fully residually free, as a consequence of Rips' Theorem. However \mathbb{Z}^n -free groups are not typically residually free: the class of \mathbb{Z}^n -free groups is thus considerably larger than the class of fully residually free groups.

H. Bass [2] examined group actions on $\mathbb{Z} \times \Lambda_0$ -trees, which in principle makes group actions on \mathbb{Z}^n -trees tractable. Recently, O. Kharlampovich, A. Myasnikov, V. Remeslennikov and D. Serbin have given a detailed description of \mathbb{Z}^n -free groups [16] (see also [17] and [18]). Moreover they have shown that in the finitely presented case, if a group G admits a regular free action on a Λ -tree for some ordered abelian group Λ , then G admits a regular free action on an \mathbb{R}^n -tree for some n .

A new ingredient to the theory was introduced into the theory by Myasnikov, Remeslennikov and Serbin, namely infinite Λ -words in a group. They showed that a group that admits a faithful representation by infinite Λ -words also admits a free action on a Λ -tree [24]. Conversely, a result of Chiswell [8] showed that a group equipped with a free action on a Λ -tree admits a faithful representation as infinite Λ -words.

Of course the appearance of generalised words in the theory should come as no surprise since the prototypical example of a free action on a $(\mathbb{Z}-)$ tree is the action of a free group on its Cayley graph (with respect to a free generating set). The Lyndon lengths of elements of the group with respect to this action coincide with the lengths as words in the free generating set. A similar statement is true of the general case.

Further progress has been made by A. Nikolaev and Serbin [25] who have given an effective solution of the membership problem and the power problem for groups that act freely on \mathbb{Z}^n -trees.

The surge in interest in actions on Λ -trees in recent times follows the work of Kharlampovich and Myasnikov on the Tarski problems concerning the elementary theory of non-abelian free groups. They showed among many other things that any pair of non-abelian free groups are elementarily equivalent, and that the elementary theory of such groups is decidable. These are also called limit groups, a term introduced by Z. Sela [29] who in a series of papers gave another solution of Tarski's problems using a different approach.

A. Martino and the author surveyed the topic of free actions on \mathbb{Z}^n -trees in [23] though this predates many of the developments in the theory outlined above.

A further advance was obtained by V. Guirardel [13] who investigated group actions on \mathbb{R}^n -trees, and showed that \mathbb{R}^n -free groups admit a graph of groups decomposition with \mathbb{R}^{n-1} -free vertex groups. He also showed that such groups are coherent, thereby giving a proof of the finite presentability of limit groups.

All actions referred to so far are isometric actions. Continuous actions on \mathbb{R} -trees by non-nesting automorphisms have also been considered by G. Levitt [19], and the more general situation of non-nesting actions on pretrees was studied by B. Bowditch and J. Crisp [6]. In particular the latter have shown that a finitely presented group that admits a non-trivial archimedean action on a median pretree also admits a continuous non-trivial non-nesting action on an \mathbb{R} -tree. Thus by the main result of [19], such a group admits a non-trivial isometric action on an \mathbb{R} -tree.

In [20], Lioussé initiated the study of affine actions on \mathbb{R} -trees. An affine automorphism of an $(\mathbb{R}-)$ metric space is a surjective function $\phi : X \rightarrow X$ for which there exists a constant α_ϕ such that $d(\phi x, \phi y) = \alpha_\phi d(x, y)$ for all $x, y \in X$. (Of course such an α_ϕ must be positive for non-degenerate X .) Examples of affine actions include the following, due to F. Paulin [28]. Suppose that Γ is a hyperbolic group, and H a subgroup of $\text{Aut}(\Gamma)$ with amenable image in $\text{Out}(\Gamma)$ and with infinite centre. Then $\Gamma \rtimes H$ admits an affine action on an \mathbb{R} -tree such that the restriction to Γ is isometric, has no global fixed point, and has virtually cyclic arc stabilisers.

Lioussé gave two families of examples of groups (recalled as Example 2.1 below) that admit free affine actions on \mathbb{R} -trees but which admit no free isometric action on an \mathbb{R} -tree. Martino and the author [22] later showed that Lioussé's groups do admit free isometric actions on \mathbb{Z}^n -trees for some n . This work may be thought of as an attempt to admit affine actions on \mathbb{R} -trees to the fold of

isometric actions on Λ -trees at the expense of modifying the ordered abelian group. However, it is also natural to reverse this process — that is, to attempt to broaden the scope of affine actions on \mathbb{R} -trees to incorporate non-archimedean Λ .

In [26] the author considered the problem of equivariantly embedding a pretree in a Λ -tree equipped with an isometric action, in the presence of a group action on the former. One necessary condition for such an embedding to exist is *rigidity*: a map g is rigid if $g[x, y] \not\subset [x, y]$ and $g[x, y] \not\supset [x, y]$ for all points x and y . However it was shown that this condition is not sufficient as the wreath product $C_\infty \wr C_\infty$ of two infinite cyclic groups admits a free, rigid action on a pretree but does not admit a free isometric action on any Λ -tree. From the point of view of isometric actions, $C_\infty \wr C_\infty$ may thus appear to be somewhat pathological.

However, the concept of affine action can be naturally extended to Λ -trees. The key idea here is to require $d(gx, gy) = \alpha_g d(x, y)$ where now α_g is an element of $\text{Aut}^+(\Lambda)$, the group of order-preserving group automorphisms of Λ . In this framework, isometric actions on Λ -trees and affine actions on \mathbb{R} -trees in the sense of Lioussé appear as special cases. Moreover, examples such as the action of $C_\infty \wr C_\infty$ mentioned above — and many others besides — fall under this new heading of affine action on a Λ -tree. The examples presented here may be summarised as follows.

- Theorem 0.1.** (i). *The Heisenberg group $\text{UT}(3, \mathbb{Z})$ admits a free affine action on a \mathbb{Z}^3 -tree.*
(ii). *The wreath product of two infinite cyclic groups admits a free affine action on a $\mathbb{Z} \times \mathbb{R}$ -tree.*
(iii). *The soluble Baumslag-Solitar groups $\text{BS}(1, a)$ admit a free affine action on a $\mathbb{Z} \times \mathbb{R}$ -tree.*

The main goal of the current paper is to commence a systematic study of affine actions on Λ -trees X , where Λ is not necessarily archimedean — the titular non-archimedean trees. We will focus especially on constructions that yield free affine actions. In future work, we propose to focus on the case $\Lambda = \mathbb{Z}^n$, and on the case where X is a linear tree. Various technical difficulties present themselves, notably the absence of a good analogue of hyperbolic length function. Nevertheless much of our work follows the general programme of the theory of isometric actions. For example, we show that the duality between Lyndon length functions and isometric actions on Λ -trees, which was first proven by Chiswell [7] in the case $\Lambda = \mathbb{Z}$ extends naturally to the affine case.

Theorem 0.2. *Let $\alpha : G \rightarrow \text{Aut}^+(\Lambda)$ be a homomorphism, fix an α -affine action of G on a Λ -tree X , and a point $x \in X$. The function defined by $L(g) = L_x(g) = d(x, gx)$ is an α -affine Lyndon length function on G .*

Conversely, for any α -affine Lyndon length function L on G there is an (essentially unique) α -affine action of G on a Λ -tree, and a basepoint x such that $L = L_x$.

We also generalise some of Lioussé's results, for example showing that under certain conditions an isometric action of a normal subgroup of G can be extended to an affine action of G . Some of the results established in the general case give rise to results that are new even in the archimedean case. For example, we show

Theorem 0.3. *The class of groups that admit a free affine action on a Λ -tree is closed under free products.*

One of the striking features of the theory of free affine actions on non-archimedean trees is that stabilisers of lines are no longer necessarily torsion-free abelian. In the case $\Lambda = \mathbb{Z}^n$, a group that stabilises a line and preserves the orientation is finitely generated torsion-free nilpotent, while for Λ with $\text{Aut}^+(\Lambda)$ soluble, the line stabilisers are torsion-free soluble, but not necessarily nilpotent. This means that certain familiar properties of isometrically 'tree-free' groups such as the CSA property must be generalised.

Recently, we have shown that groups that admit free affine actions on linear Λ -trees form a much larger class than would be predicted from the isometric case. Namely, *residually torsion-free nilpotent groups* admit free affine actions on linear Λ -trees. This is shown in [27]. It is, as far as we are aware, an open question whether groups that admit a free isometric action on a \mathbb{Z}^n -tree are residually torsion-free nilpotent.

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1 Fundamentals

1.1 Ordered abelian groups and their automorphism groups

Let Λ (or more precisely $(\Lambda, +, \leq)$) be an ordered abelian group. We write $\text{Aut}^+(\Lambda)$ for the group of all group automorphisms of Λ that preserve the order. A *convex* subgroup of Λ is a subgroup Λ_0 , for which $a, c \in \Lambda_0$ with $a \leq b \leq c$ implies $b \in \Lambda_0$. The set of convex subgroups of an ordered abelian group Λ is linearly ordered by inclusion, and any $\alpha_* \in \text{Aut}^+(\Lambda)$ induces an order-preserving permutation of the convex subgroups of Λ . We call an automorphism *shift-free* if this induced permutation is trivial — that is, if it stabilises each convex subgroup. We write $\text{Aut}_0^+(\Lambda)$ for the group of shift-free automorphisms of Λ . We will call an ordered abelian group *convex-rigid* if all order-preserving automorphisms are shift-free.

The order type of the set of non-zero convex subgroups of Λ is the *rank* of Λ . If Λ has finite rank n then Λ embeds (as an ordered abelian group) in \mathbb{R}^n — here $\Lambda_1 \times \Lambda_2$ is ordered lexicographically via $(\lambda_1, \lambda_2) \leq (\lambda'_1, \lambda'_2)$ if $\lambda_1 < \lambda'_1$ or if $\lambda_1 = \lambda'_1$ and $\lambda_2 \leq \lambda'_2$. Inductively this gives an order on $\Lambda_1 \times \cdots \times \Lambda_n$ and, as a special case, on Λ_0^n , where the Λ_i are given ordered abelian groups. Whenever an ordered abelian group is given as a direct decomposition or as a (direct) power, it is always the lexicographic order that we will consider unless the contrary is specifically stated. It is worth noting however that other orders are possible. For example, $\mathbb{Z} \times \mathbb{Z}$ may be lexicographically ordered, making it a rank 2 ordered abelian group, or it may have rank 1, via an embedding in \mathbb{R} . Subgroups of \mathbb{R} (with the natural order) have rank ≤ 1 and are characterised by the familiar archimedean property.

The structure of $\text{Aut}^+(\Lambda)$ was considered in a paper by P. Conrad. For our purposes, it suffices to record the following (see [11, §1]).

Proposition 1.1. *If $\Lambda = \Lambda_1 \times \cdots \times \Lambda_n$ (with the lexicographic order), then $\text{Aut}_0^+(\Lambda)$ is isomorphic to the group of matrices of the form*

$$\begin{pmatrix} \alpha_n & h_{n(n-1)} & \cdots & h_{n1} \\ 0 & \alpha_{n-1} & \cdots & h_{(n-1)1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_1 \end{pmatrix}$$

where $\alpha_i \in \text{Aut}_0^+(\Lambda_i)$ and $h_{ij} \in \text{Hom}(\Lambda_j, \Lambda_i)$.

In particular $\text{Aut}_0^+(\Lambda_1 \times \Lambda_2) \cong (\text{Aut}_0^+(\Lambda_1) \times \text{Aut}_0^+(\Lambda_2)) \ltimes \text{Hom}(\Lambda_1, \Lambda_2)$. (Here the action of $\text{Aut}_0^+(\Lambda_1) \times \text{Aut}_0^+(\Lambda_2)$ is given by $(\alpha_1, \alpha_2) \cdot h = \alpha_2 h \alpha_1^{-1}$ for $\alpha_i \in \text{Aut}_0^+(\Lambda_i)$ and $h \in \text{Hom}(\Lambda_1, \Lambda_2)$.)

One can deduce from this result that the group $\text{Aut}^+(\mathbb{Z}^n)$ is isomorphic to the group $\text{UT}(n, \mathbb{Z})$ of upper triangular matrices with all diagonal entries equal to 1. Note also that the group $\text{T}(n, \mathbb{R})$ of upper triangular matrices with positive units on the diagonal, embeds in $\text{Aut}^+(\mathbb{R}^n)$ but is not isomorphic to it on account of the profusion of (non-order-preserving) homomorphisms $\mathbb{R} \rightarrow \mathbb{R}$.

Next we consider $\text{Aut}^+(\Lambda)$ where $\Lambda \leq \mathbb{R}$. In this case $\alpha_* \in \text{Aut}^+(\Lambda) = \text{Aut}_0^+(\Lambda)$ is determined by $\alpha_* \cdot 1$, and has the effect of multiplication by the positive real number $\alpha_* \cdot 1$. One concludes that $\text{Aut}_0^+(\Lambda)$ embeds in the multiplicative group of positive real numbers which is isomorphic to $\mathbb{R} = (\mathbb{R}, +)$.

Corollary 1.2. *Let $\Lambda = \Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_n$, suppose that Λ_i is convex-rigid and that $\text{Aut}^+(\Lambda_i)$ is soluble for all i . Then $\text{Aut}^+(\Lambda)$ is soluble. In particular, $\text{Aut}^+(\Lambda)$ is soluble for any ordered abelian group Λ of finite rank.*

Note that in general a group G admits an embedding in $\text{Aut}^+(\Lambda)$ for some Λ precisely when G is right orderable, a result due to D. Smirnov (see [5, Theorem 7.1.3]). In fact R. Göbel and S. Shelah [12] have shown that G is right orderable if and only if G is isomorphic to $\text{Aut}^+(\Lambda)$ for some Λ .

1.2 Affine length functions and actions on Λ -trees

We refer the reader to Chiswell's book [10] where the basic theory of Λ -trees is treated in detail. (We state a characterisation of Λ -trees just after Lemma 1.3 below.)

Let (X, d) and (X', d') be Λ -trees. In fact most of the definitions that follow make sense for Λ -metric spaces in general. A function $\phi : X \rightarrow X'$ is an α_* -affine map (or simply an affine map) if there exists $\alpha_* = \alpha_\phi \in \text{Aut}^+(\Lambda)$ such that

$$d'(\phi x, \phi y) = \alpha_*(d(x, y))$$

for all $x, y \in X$. The automorphism α_* is called the *dilation factor* corresponding to ϕ . Note that one can define an affine map somewhat more permissively by requiring only that $\alpha_* \in \text{Aut}^+(\Lambda_0)$ where Λ_0 is the (convex) subgroup generated by the image of the distance function d , but we will not do so here. Note however that if Λ has finite rank then $\alpha_* \in \text{Aut}^+(\Lambda_0)$ extends in a natural (but typically non-unique) way to $\bar{\alpha}_* \in \text{Aut}^+(\Lambda)$. Thus $\text{Aut}^+(\Lambda_0)$ embeds in $\text{Aut}^+(\Lambda)$ in this case.

An *affine isomorphism* is a bijective affine map, and an *affine automorphism* is an affine isomorphism with the same domain and codomain. Note that if $\Lambda = \mathbb{R}$ this usage is consistent with Lioussé [20]. An *affine action* of G on a Λ -tree X consists of a homomorphism $\alpha : G \rightarrow \text{Aut}^+(\Lambda)$, written $g \mapsto \alpha_g$, together with an action of G on X such that g is an α_g -affine automorphism of X for all $g \in G$. Such an action may be referred to as an α -affine action. Note that α_g restricts to the identity map on the image of d precisely when g is an isometry so that the subgroup of G consisting of isometries contains $\ker \alpha$.

Let G be a group that has an α -affine action on a Λ -tree X , and fix $x_0 \in X$. Define a function $L : G \rightarrow \Lambda$ by the formula

$$L(g) = L_{x_0}(g) = d(x_0, gx_0)$$

Such an L is called the (α) -based length function arising from the action (with respect to x_0).

Fix a homomorphism $\alpha : G \rightarrow \text{Aut}^+(\Lambda)$. Let $L : G \rightarrow \Lambda$ be a function, and for $g, h \in G$, we define the *ancillary functions* **a**, **b** and **c** as follows.

$$\begin{aligned} \mathbf{a}(g) &= \frac{1}{2} (L(g) + L(g^{-1}) - \alpha_{g^{-1}} L(g^2)) \\ \mathbf{b}(g) &= \frac{1}{2} ((\alpha_{g^{-1}} + 1)L(g^2) - (\alpha_{g^2} + 1)L(g^{-1})) \\ \mathbf{c}(g, h) &= \frac{1}{2} (L(g) + L(h) - \alpha_g L(g^{-1}h)) \end{aligned}$$

(Here $1 + \alpha_\gamma$ is of course the map $\lambda \mapsto \lambda + \alpha_\gamma(\lambda)$.)

Call L an α -Lyndon length function on G if for g, h and k in G one has

$$(L1) \quad \mathbf{c}(g, h) \in \Lambda$$

$$(L2) \quad L(1) = 0$$

$$(L3) \quad L(g) = \alpha_g(L(g^{-1}))$$

$$(L4) \quad (\mathbf{c}(g, h), \mathbf{c}(g, k), \mathbf{c}(h, k)) \text{ is an isosceles triple — that is, } \mathbf{c}(g, h) < \mathbf{c}(g, k) \Rightarrow \mathbf{c}(g, h) = \mathbf{c}(h, k).$$

If $\alpha_g = 1$ for all g then an α -Lyndon length function is a Lyndon length function in the usual sense (apart from (L1), which is an extra assumption in, for example, [10, §2.4]).

If L is a based length function arising from an α -affine action, then applying the definition of **c** to L , one easily sees that $\mathbf{c}(g, h)$ is equal to the length of the segment $[x, v] = [x, gx] \cap [x, hx]$, as in the isometric case. Property (L4) of α -Lyndon length functions follows, while properties (L1)-(L3) are immediate. A based length function arising from an α -affine action is therefore an α -Lyndon length function. (Moreover $\mathbf{b}(g) = \ell(g) = \min\{d(x, gx) : x \in X\}$ for $g \in \ker \alpha$ if **b** arises from a based length function and g is not an inversion.) We will shortly prove the converse.

Lemma 1.3. *Let L be an α -Lyndon length function on G . Then for all $g, h \in G$ we have*

- (i). $L(g) \geq 0$;
- (ii). $L(gh) \leq L(g) + \alpha_g L(h)$;
- (iii). for $g_1, g_2, \dots, g_n \in G$, if $\bar{g}_0 = 1$ and $\bar{g}_k = \overline{g_{k-1}}g_k$ for $1 \leq k \leq n$ then $L(g_1 g_2 \dots g_n) \leq \sum_{k=1}^n \alpha_{\bar{g}_{k-1}} L(g_k)$;
- (iv). $\mathbf{c}(g, h) \geq 0$;
- (v). $\mathbf{c}(g, g) = L(g)$;
- (vi). $\mathbf{c}(g, 1) = 0$;
- (vii). $\alpha_g L(g^{-1}h) = \alpha_h L(h^{-1}g)$;
- (viii). $\mathbf{c}(g, h) = \mathbf{c}(h, g) \leq \min\{L(g), L(h)\}$;

Proof: It is clear from the definition of \mathbf{c} and of α -Lyndon length functions that $\mathbf{c}(g, g) = L(g)$ and $\mathbf{c}(g, 1) = 0$. One has $\alpha_h L(h^{-1}g) = \alpha_g \alpha_{g^{-1}h} L(h^{-1}g) = \alpha_g L(g^{-1}h)$ by (L3) whence $\mathbf{c}(g, h) = \mathbf{c}(h, g)$. Taking $k = 1$ in the isosceles condition now gives $\mathbf{c}(g, h) \geq 0$. It follows that $L(g) = \mathbf{c}(g, g) \geq 0$. Taking $g = k$ in the isosceles condition gives $\mathbf{c}(g, h) \leq L(g)$; consequently $\mathbf{c}(g, h) = \mathbf{c}(h, g) \leq L(h)$.

Now $0 \leq 2\mathbf{c}(g^{-1}, h) = L(g^{-1}) + L(h) - \alpha_{g^{-1}} L(gh)$, giving $L(g) - L(gh) + \alpha_g L(h) = \alpha_g (L(g^{-1}) + L(h) - \alpha_{g^{-1}} L(gh)) \geq 0$, whence $L(gh) \leq L(g) + \alpha_g L(h)$. The inequality in (3) follows by induction on n . \square

Recall that a Λ -metric space (X, d) is δ -hyperbolic (with respect to v) if $(x \cdot y) \geq \min\{(x \cdot z), (y \cdot z)\} - \delta$ for $x, y, z \in X$: here $(x \cdot y) = (x \cdot y)_v = \frac{1}{2}(d(x, v) + d(y, v) - d(x, y))$. Indeed, this ‘Gromov inner product’ gives a convenient characterisation of Λ -trees which we will use later: a Λ -metric space is a Λ -tree if and only if, for all $x, y, v \in X$ we have

- (i). $(x \cdot y)_v \in \Lambda$;
- (ii). X is geodesic; and
- (iii). X is 0-hyperbolic (with respect to v).

In fact it is sufficient to check these conditions for a particular choice of v while x and y vary.

Three points of a Λ -tree X are *collinear* if there is a segment in X containing all three. A Λ -tree is *linear* if x, y and z are collinear for all $x, y, z \in X$. Equivalently X is linear if it admits an isometric embedding in Λ (see [10, §2.3]).

We write $[x, y, z]$ as a common shorthand for the segment $[x, z]$ where $y \in [x, z]$, and for the assertion that $y \in [x, z]$. More generally, $[x_1, \dots, x_n]$ denotes the segment $[x_1, x_n]$ where

$$d(x_1, x_m) = \sum_{k=1}^m d(x_{k-1}, x_k) \text{ for all } 1 \leq m \leq n$$

and the assertion that the displayed equation holds. If $x_{i-1} \neq x_i$ we may replace the comma between them by a semicolon. For example $[x, y, z; w]$ implies that $y \in [x, z]$ and $z \in [x, w]$, and that $z \neq w$. This of course implies that $x \neq w$ and $y \neq w$ also, and that $z \in [y, w]$.

In general the segments $[x, y]$, $[x, z]$ and $[y, z]$ have exactly one point in common, which is denoted $Y(x, y, z)$ (see [10, 2.1.2]).

Lemma 1.4. (i). Let ϕ be an α_* -affine isomorphism $X \rightarrow X'$ where X and X' are Λ -trees. Then $y \in [x, z] \Leftrightarrow \phi(y) \in [\phi(x), \phi(z)]$ and $u = Y(x, y, z) \Leftrightarrow \phi(u) = Y(\phi(x), \phi(y), \phi(z))$.

(ii). Suppose that $\phi : X \rightarrow X'$ is a function, $v \in X$ and that for linear subtrees J of X with v as an endpoint the restriction $\phi|_J$ is α_* -affine. If $\phi(Y(x, y, v)) = Y(\phi(x), \phi(y), \phi(v))$ then ϕ is α_* -affine.

Proof: Note that $y \in [x, z]$ if and only if $d(x, y) + d(y, z) = d(x, z)$. Applying α_* to both sides of this equation gives the required implications for the first assertion of (1). For the second assertion, note that u is characterised by the fact that it belongs to the three segments $[x, y]$, $[x, z]$ and $[y, z]$. By the first assertion, $\phi(u)$ belongs to the images of these segments under ϕ , giving the result.

To establish (2), for $x, y \in X$, we put $u = Y(x, y, v)$. Then

$$\begin{aligned} d(\phi(x), \phi(y)) &= d(\phi(x), \phi(u)) + d(\phi(u), \phi(y)) \\ &= \alpha_* d(x, u) + \alpha_* d(u, y) \\ &= \alpha_* d(x, y). \end{aligned}$$

□

Theorem 1.5. (i). Let (X, d) be a Λ -metric space and $v \in X$ such that

- (a) $(x \cdot y)_v \in \Lambda$;
- (b) X is 0-hyperbolic with respect to v .

Then there exists a Λ -tree (X', d') and an isometric embedding $\phi : X \rightarrow X'$ such that if $\psi : X \rightarrow Z$ is any α_* -affine map (where Z is a Λ -tree) then there is a unique α_* -affine map $\mu : X' \rightarrow Z$ such that $\mu \circ \phi = \psi$.

- (ii). In the notation of part (1), an α -affine action on the Λ -metric space X has a unique extension to an α -affine action on the Λ -tree X' .
- (iii). Let (X_1, d_1) be a Λ_1 -tree, $\alpha : G \rightarrow \text{Aut}^+(\Lambda_1)$ a homomorphism, let an α -affine action of G on X_1 be given and let $h : \Lambda_1 \rightarrow \Lambda_2$ be an order preserving homomorphism. Suppose that $\bar{\alpha} : G \rightarrow \text{Aut}^+(\Lambda_2)$ is a homomorphism such that $h \circ \alpha_g = \bar{\alpha}_g \circ h$ for all $g \in G$. Then there is a Λ_2 -tree (X_2, d_2) , an $\bar{\alpha}$ -affine action of G on X_2 and a map $\phi : X_1 \rightarrow X_2$ satisfying

- (a) $d_2(\phi(x), \phi(y)) = h d_1(x, y)$ for $x, y \in X_1$;
- (b) $\phi(gx) = g\phi(x)$ for $x \in X_1$.
- (c) for $p \in X_1$, we have $L_2(g) = h L_1(g)$ where L_1 denotes the based length function with respect to p and L_2 the based length function with respect to $\phi(p)$.

Proof: (1) The Λ' -tree (X', d') is as constructed in [10, Theorem 2.4.4]. We will follow the notation used there. To prove our assertion, it suffices to show that a suitable map μ can be found for a given ψ . Obviously such a map must satisfy $\mu\langle x, d(x, v) \rangle = \mu(\phi(x)) = \psi(x)$. Moreover, for $x \in X$ and $m \geq 0$ in Λ with $m \leq d(x, v)$, if we let x_m be the point on the segment $[\psi(v), \psi(x)]_{X'}$ at distance $\alpha_* m$ from $x_0 = \psi(v)$, then any μ with the required properties must satisfy $\mu\langle x, m \rangle = x_m$. This amounts to a definition of μ , and it remains only to show that μ is α_* -affine. Now for $n \leq m \leq d(x, v)$ we have

$$\begin{aligned} d_Z(\mu\langle x, m \rangle, \mu\langle x, n \rangle) &= d_Z(x_m, x_n) \\ &= d_Z(x_m, x_0) - d_Z(x_n, x_0) \\ &= \alpha_* m - \alpha_* n \\ &= \alpha_*(m - n) \\ &= \alpha_* d_{X'}(\langle x, m \rangle, \langle x, n \rangle). \end{aligned}$$

So μ restricted to subtrees J with v as an endpoint is therefore α_* -affine. The result now follows from Lemma 1.4(2).

(2) Fix an α -affine action of G on X . Let $Z = X'$, for $g \in G$ let $\psi = \phi \circ g$, and let μ_g be the associated map $X' \rightarrow X'$. We claim that the assignation $g \cdot x_m = \mu_g(x_m)$ defines the required α -affine action of G on X' . That each μ_g is α_g -affine follows from part (1).

Now note that $\mu_g \circ \phi = \phi \circ g$, whence

$$\begin{aligned} (\mu_g \circ \mu_h) \circ \phi &= \mu_g \circ \phi \circ h \\ &= \phi \circ (g \circ h), \end{aligned}$$

So μ_{gh} and $\mu_g \circ \mu_h$ both satisfy $\mu \circ \phi = \phi \circ (g \circ h)$. Since such a map is unique by part (1), we find that $\mu_{gh} = \mu_g \circ \mu_h$. It is also easy to see that $\mu_1 = \text{id}_{X'}$, giving $\text{id}_{X'} = \mu_{g^{-1}} \circ \mu_g = \mu_g \circ \mu_{g^{-1}}$. This shows that μ_g and $\mu_{g^{-1}}$ are mutually inverse functions on X' . In particular, they are bijections.

(3) The Λ_2 -tree (X_2, d_2) is the the base change functor as described in [10, 2.4.7], where property (a) is shown. A 0-hyperbolic Λ_2 -metric space Z is constructed, whose points are equivalence classes $\langle x \rangle$ of points of X_1 under the relation $x \sim y$ if $h d_1(x, y) = 0$. We put $\phi(x) = \langle x \rangle$. It remains to show that the natural action of G on X_2 ($g\langle x \rangle = \langle gx \rangle$) is $\bar{\alpha}$ -affine. (Part (b) is then clear, and the last part is routine.) So observe that

$$\begin{aligned} d_2(\langle gx \rangle, \langle gy \rangle) &= h d_1(gx, gy) \\ &= h \alpha_g d_1(x, y) \\ &= \bar{\alpha}_g h d_1(x, y) \\ &= \bar{\alpha}_g d_2(\langle x \rangle, \langle y \rangle). \end{aligned}$$

□

The Λ_2 -tree X_2 of Theorem 1.5(3) is called the *base change functor* and is denoted $\Lambda_2 \otimes_{\Lambda_1} X_1$. (Note however that it depends on h .)

The existence of a homomorphism $\bar{\alpha}$ as in Theorem 1.5 implies that $\ker(h)$ is α_g -invariant for all g . If h is surjective, this condition is necessary and sufficient. Otherwise, this condition ensures that a suitable $h(\Lambda_1)$ -tree can be found, and the problem reduces to extending the automorphisms $\bar{\alpha}_g$ of $h(\Lambda_1)$ to automorphisms of Λ_2 . However extending automorphisms in this way is not possible in general: for example, $\text{Aut}^+(\langle \mathbb{Q}, \sqrt{2} \rangle)$ is trivial, while $\text{Aut}^+(\mathbb{Q})$ is not.

As in the isometric case, an important special case arises when we take $\Lambda_1 = \Lambda_2$ and $h(\lambda) = 2\lambda$. The resulting Λ -tree is called the *barycentric subdivision* of X . Taking $\bar{\alpha} = \alpha$, it is easy to see that the hypotheses of the theorem are satisfied, so an affine action on X extends to an affine action on the barycentric subdivision in general. As in the isometric case, if a segment $[x, y]$ is stabilised by g then there is a point of the barycentric subdivision fixed by g , namely the midpoint of the segment.

Theorem 1.6. *Let Λ be an ordered abelian group, G a group and $\alpha : G \rightarrow \text{Aut}^+(\Lambda)$ a homomorphism.*

If L is a Λ -valued α -Lyndon length function on G there exists a Λ -tree X on which G has an α -affine action that induces L .

Proof: Suppose that a Λ -valued α -Lyndon length function L is given. Define $\delta : G \times G \rightarrow \Lambda$ by the rule $\delta(g, h) = \alpha_g L(g^{-1}h)$. It follows from Lemma 1.3(7) that δ is symmetric. Moreover, by Lemma 1.3(ii) we have $L(g^{-1}k) \leq L(g^{-1}h) + \alpha_{g^{-1}h} L(h^{-1}k)$. Applying α_g to both sides of this inequality yields $\delta(g, k) \leq \delta(g, h) + \delta(h, k)$. We also have $\delta(\gamma g, \gamma h) = \alpha_\gamma \delta(g, h)$ and $\delta(g, g) = 0$, whence (G, δ) is a pseudometric space. Denoting the corresponding metric space by $(\bar{G}, \bar{\delta})$, we also see that right multiplication by G gives an α -affine action. Taking $\bar{1}$ to be the basepoint for the Gromov inner product on \bar{G} , it is routine to show that $(\bar{g} \cdot \bar{h})_{\bar{1}} = \mathbf{c}(g, h)$. Thus $(\bar{g} \cdot \bar{h})_{\bar{1}} \in \Lambda$.

To show that $(\bar{G}, \bar{\delta})$ is a 0-hyperbolic Λ_0 -metric space with respect to $\bar{1}$, it suffices to show that for all $\bar{g}, \bar{h}, \bar{k} \in \bar{G}$ we have the implication

$$\begin{aligned} \bar{\delta}(\bar{g}, \bar{1}) + \bar{\delta}(\bar{h}, \bar{1}) - \bar{\delta}(\bar{g}, \bar{h}) &< \bar{\delta}(\bar{g}, \bar{1}) + \bar{\delta}(\bar{k}, \bar{1}) - \bar{\delta}(\bar{g}, \bar{k}) \\ \Rightarrow \bar{\delta}(\bar{g}, \bar{1}) + \bar{\delta}(\bar{h}, \bar{1}) - \bar{\delta}(\bar{g}, \bar{h}) &= \bar{\delta}(\bar{h}, \bar{1}) + \bar{\delta}(\bar{k}, \bar{1}) - \bar{\delta}(\bar{h}, \bar{k}). \end{aligned}$$

This follows from the isosceles condition on \mathbf{c} .

So G has an α -affine action on the 0-hyperbolic Λ_0 -metric space \bar{G} with $\bar{\delta}(\gamma \bar{1}, \bar{1}) = L(\gamma)$. By Theorem 1.5, \bar{G} has a G -equivariant embedding in a Λ -tree with the required properties. □

As in the isometric case, the Λ -tree X in Theorem 1.6 is unique in the following sense. If (Z, d'') is a Λ -tree on which G has an α -affine action, and $w \in Z$ satisfies $L = L_w$ then there is a unique G -equivariant isometry $\mu : X \rightarrow Z$ such that $\mu(x) = w$ and the image of μ coincides with the subtree of Z spanned by the orbit Gw . The proof of this assertion closely follows the proof in the isometric case (see [10, Theorem 4.6]) and will be omitted.

In the literature a group is said to be Λ -free if it admits a free isometric action (without inversions) on a Λ -tree, and tree-free if it is Λ -free for some Λ . We will say that a group is ITF

(isometric tree-free) if it admits a free isometric action without inversions on a Λ -tree for some Λ . A group is ATF (affine tree-free) if it admits a free affine action without inversions on a Λ -tree for some Λ . If we want to specify the ordered abelian group we will refer instead to an ITF(Λ) or ATF(Λ) group as appropriate. Note that henceforth a free action will be assumed to be without inversions unless the contrary is explicitly stated.

1.3 Affine automorphisms of Λ -trees

Let g be an affine automorphism of a Λ -tree X . Let X^g denote the set of points of X fixed by g . Note that if $\alpha_g \neq 1 = \text{id}_\Lambda$ the set of all fixed points need *not* form a subtree of X , as the map $(x, y, z) \mapsto (x, y, z + y + x)$ in the case $X = \Lambda = \mathbb{Z}^3$ shows. Let $A_g = \{x \in X : x \in [g^{-1}x, gx]\}$, and $\tilde{A}_g = \{x \in X : g^{-1}x, x, gx \text{ are collinear}\}$. Call g *elliptic* if $X^g \neq \emptyset$. Note that $X^g \subseteq X^{g^n}$ and $A_g \subseteq A_{g^n}$ for $n \neq 0$, and $A_g \subseteq \tilde{A}_g$.

We call g an *inversion* if $X^g = \emptyset$ but $X^{g^2} \neq \emptyset$. In this case there exists a segment $[x, y]$ in X whose endpoints are swapped by g , and g^2 stabilises the segment. Note that like in the isometric case, if g is an inversion of X then g has a unique fixed point in the barycentric subdivision.

Call g a *nesting reflection* if $X^{g^2} = \emptyset$ and there exists $x \in X$ such that $[x; gx; g^{-1}x]$ or $[x; g^{-1}x; gx]$. In this case g^n fixes no point for $n \neq 0$, and either $g[x, gx] \subset [x, gx]$ or $g[x, gx] \supset [x, gx]$. See Example 1.9 for an example of this behaviour.

If g does not satisfy any of the criteria above, then g is said to be *hyperbolic*. As we will see (Theorem 1.11), a hyperbolic affine automorphism behaves much like its isometric counterpart.

Let us clarify the situation that arises if there is some point $x \in \tilde{A}_g \setminus A_g$ — a situation that cannot arise in the isometric case unless g is an inversion.

Suppose that g is an affine automorphism of X , and that $x \in X$ satisfies $[x; gx, g^{-1}x]$ or $[x; g^{-1}x, gx]$. We lose no generality in the following discussion by assuming the former. Suppose further that $y \in A_g$ — thus $y \in X$ satisfies $[g^{-1}y, y, gy]$. Suppose initially that y satisfies $[y, g^n x, g^{n+1}x]$ for some $n \in \mathbb{Z}$. Then replacing y by $g^{\pm 1}y$ if necessary, we can ensure that $[g^{-1}y, y, gy, g^n x; g^{n+1}x, g^{n-1}x]$ or $[gy, y, g^{-1}y, g^n x; g^{n+1}x, g^{n-1}x]$. But in either case we get a contradiction: in the first case, we have $g[g^n x, y] \not\subseteq [g^n x, y]$ contradicting $g^{-1}[g^n x, y] \supset [g^n x, y]$; while in the second case, we obtain $g[g^n x, y] \supset [g^n x, y]$ which contradicts $g^{-1}[g^n x, y] \not\subseteq [g^n x, y]$.

Therefore for any $y \in X$ satisfying $[g^{-1}y, y, gy]$ we must have $v = Y(y, g^n x, g^{n+1}x) \notin \{g^n x, g^{n+1}x\}$ for all $n \in \mathbb{Z}$ and thus $v \in I = \cap_{n \in \mathbb{Z}} [g^n x, g^{n+1}x]$. If $y \in I$ (so $y = v$) and $gy \neq y$ then (replacing x by gx if necessary) we have $[g^n x, g^{n+2}x, \dots, g^{-1}y; y; gy, \dots, g^{n+1}x, g^{n-1}x] \subseteq I$, which forces a contradiction; thus $gy = y$. This shows that $A_g \cap I$ consists of points fixed by g (though it may well be empty).

Let us show next that g can fix at most one point of $[x, gx]$. For if $p, q \in [x, gx]$ are both fixed by g then $p, q \in I$ and we have $[x, g^2x, \dots, p, q, \dots, g^3x, gx]$ which implies $[g^{-1}x, gx, \dots, g^{-1}p, g^{-1}q, \dots, g^2x, x]$. Thus we have both $[x, p, q, gx]$ and $[x, q, p, gx]$, giving $q = p$.

Finally, note that for any y such that $[g^{-1}y, y, gy]$ holds, if w belongs to the segment $[g^{-1}y, y, gy]$ and to $[x, gx]$ then $[g^{-1}w, w, gw]$ holds also, so we can replace y by w in the argument above to conclude that $gw = w$, and hence $g^{-1}y = y = gy$. Otherwise $[g^{-1}y, y, gy]$ has trivial intersection with $[x, gx]$, and $v = Y(g^\epsilon y, g^n x, g^{n+1}x)$ for $\epsilon \in \{-1, 0, 1\}$. We claim that v is fixed by g (and is thus the unique fixed point of g in the segment $[x, gx]$). For $[g^n x; g^{n+1}x, g^{n-1}x]$ holds, which implies

$$\begin{aligned} v &= Y(g^{-1}y, g^{n-1}x, g^n x) \\ &= g^{-1}Y(y, g^n x, g^{n+1}x) \\ &= g^{-1}v. \end{aligned}$$

In summary,

Lemma 1.7. *Suppose that $[x; gx, g^{-1}x]$ or $[x; g^{-1}x, gx]$ holds.*

- (i). *Then g has at most one fixed point v in $[x, gx]$, in which case $v \in \cap_{n \in \mathbb{Z}} [g^n x, g^{n+1}x]$ and any other fixed point y in X must satisfy $v = Y(y, x, gx)$.*

- (ii). If $y \in A_g$ then $Y(y, gx, x)$ is a fixed point of g . If g is not elliptic then $A_g = \emptyset$.
- (iii). If g^2 is not elliptic then g is not an inversion; thus g is a nesting reflection.
- (iv). If h is an affine automorphism of X with $A_h \neq \emptyset$ and h fixes no point then h is hyperbolic.
- (v). If h is a nesting reflection then h^2 is hyperbolic.
- (vi). If h is hyperbolic, then h^n is hyperbolic for all $n \neq 0$. □

The case where g is hyperbolic is described further in the next theorem. For now let us warn the reader that an affine automorphism may behave near one point like an elliptic automorphism, and elsewhere like a hyperbolic automorphism. Moreover, unlike the isometric case, it is not always possible to distinguish between the cases we describe by means of based length functions.

Example 1.8. Fix a real number $0 < a < 1$ and let X be the quotient of $[0, \infty)_{\mathbb{R}} \times \{1, 2, 3\}$, where $(0, i)$ is identified with $(0, j)$ for all i and j , and put

$$d((x, i), (y, j)) = \begin{cases} |x - y| & \text{if } i = j \\ |x| + |y| & \text{otherwise} \end{cases}$$

Define $g : X \rightarrow X$ via

$$g : (x, i) \mapsto \begin{cases} (ax, 3 - i) & i \in \{1, 2\} \\ (ax, 3) & i = 3. \end{cases}$$

Then g is α_* -affine where $\alpha_*x = ax$. Since (the equivalence class of) $(0, 1)$ is fixed by g , the latter is clearly elliptic. However, taking X_3 to be the subtree consisting of points of the form $(x, 3)$ with $x \neq 0$, we see that X_3 is g -invariant and the restriction of g to this subtree is hyperbolic.

On the other hand, taking X_{12} to be the complement of X_3 , we see that this is also a g -invariant subtree and that $[p; gp; g^{-1}p]$ holds for all $p \neq (0, i)$. Thus (denoting the restriction of g to X_{12} also by g) we have $A_g = X_{12}$ while A_g consists of a single point, which is fixed by g . It is easy to show that $\mathbf{b}_p(g) < 0$ for all $p \in X_{12}$ where p is not the fixed point, while $\mathbf{b}_p(g) > 0$ for $p \in X_3$. Of course $\mathbf{b}_p(g) = 0$ precisely when p is the fixed point. (See §1.2 for the definition of \mathbf{b} .)

Example 1.9. Let us also describe an example of a nesting reflection. Let $X = \Lambda = \mathbb{R}$. Put $g \cdot x = 2 - \frac{x}{2}$. Then the unique fixed point of g^2 is $4/3$, which is also the unique fixed point of g . However if we replace $\Lambda = \mathbb{R}$ by the subgroup Λ_0 consisting of the dyadic rationals $(a/2^n)$, then g stabilises Λ_0 , but g and g^2 have no fixed point, since $4/3$ is not an element of Λ_0 .

However, if automorphisms are assumed to be rigid, the situation becomes easier to analyse. An automorphism g of a Λ -tree is *rigid* (or *non-nesting*) if no closed segment is mapped properly into itself by g or g^{-1} . In this case, nesting reflections described above are not possible (as the nomenclature suggests), and if g fixes a point, the set of all fixed points forms a subtree. If g is rigid and g^2 does not fix a point, then g is hyperbolic in a similar sense to the isometric case: there is a maximal linear g -invariant subtree A_g , such that $x < gx$ for all $x \in A_g$ (with respect to one of the natural linear orders on A_g).

Note that if g is hyperbolic and A_g is spanned by a single $\langle g \rangle$ -orbit then g is rigid. (Such a g is said to be archimedean.)

In [14], Hudson considers rigid group actions on pretrees, and shows that much of the theory of isometries and of isometric actions can be extended to this general setting.

Since the rigid case is better behaved than the general case, we will focus much of our attention on the rigid case, and show where possible that rigidity is preserved by our constructions. We make no such restriction in the next theorem however. First, a lemma.

Lemma 1.10. *If I_1 and I_2 are disjoint linear subtrees of X , then either $I_1 \cup I_2$ is collinear or, for $i = 1$ or $i = 2$, there exists $x_0 \in I_i$ with $[x_0, z] \cap I_i = \{x_0\}$ for all $z \in I_{3-i}$. The point x_0 belongs to every segment joining a point of I_1 to a point of I_2 .*

Proof: Assume that $I_1 \cup I_2$ is not collinear. Then there exist $x, y, z \in I_1 \cup I_2$ with $u = Y(x, y, z) \notin \{x, y, z\}$. Without loss of generality, $x, y \in I_1$ and $z \in I_2$. We claim that $i = 1$ and $x_0 = u$ give the required properties. Firstly, suppose that $v \in [u, z] \cap I_1$. Then we have $[z; v, u, x]$ and $[z; v, u, y]$, so that $u = Y(x, y, z) = Y(x, y, v)$. If $u \neq v$ this contradicts the linearity of I_1 , since $x, y, v \in I_1$. This shows that $[u, z] \cap I_1 = \{u\}$. Now if z' is any point of I_2 , we have

$$\begin{aligned} [u, z'] \cap I_1 &\subseteq ([u, z] \cap I_1) \cup ([z, z'] \cap I_1) \\ &= \{u\} \cup \emptyset. \end{aligned}$$

Now suppose that $s \in I_1$ and $t \in I_2$; we claim that $x_0 = u \in [s, t]$. Put $w = Y(s, t, u)$ and observe that $w \in I_1$, since $s, u \in I_1$. Thus

$$\begin{aligned} w &\in [w, t] \cap I_1 \\ &\subseteq [u, t] \cap I_1 \\ &= \{u\}, \end{aligned}$$

forcing $u = w \in [s, t]$. □

Note that for any affine automorphism g we have the implications

$$\begin{aligned} u \in A_g &\Leftrightarrow [g^{-1}u, u, gu] \\ &\Leftrightarrow [\gamma g^{-1}u, \gamma u, \gamma gu] \\ &\Leftrightarrow [(\gamma g^{-1}\gamma^{-1})(\gamma u), \gamma u, (\gamma g\gamma^{-1})(\gamma u)] \\ &\Leftrightarrow \gamma u \in A_{\gamma g\gamma^{-1}} \end{aligned}$$

Thus $\gamma \cdot A_g = A_{\gamma g\gamma^{-1}}$. Similarly $\gamma \cdot \tilde{A}_g = \tilde{A}_{\gamma g\gamma^{-1}}$.

If I and J are disjoint subtrees of X , we use the notation $\overline{\text{Br}}(I, J)$ for the (closed) bridge between I and J (see [10, §2.1]). We will also write $\text{Br}^o(I, J)$ for the open bridge between I and J : this is the set $[x, y] \setminus (I \cup J)$ where $x \in I$ and $y \in J$, and can be shown to be a linear subtree independent of these points.

The following theorem follows closely the argument given in [10, 3.1.4] for the isometric case, but is in fact valid in the general situation where X is a median pretree and g is any pretree automorphism. (except of course for the last two parts which refer to d).

Theorem 1.11. *Suppose that g is hyperbolic (that is, suppose that g fixes no point of X and is not a nesting reflection or an inversion). For $x \in X$, we put $u = u_x = u(x, g) = Y(g^{-1}x, x, gx)$.*

Then

- (i). A_g is non-empty;
- (ii). A_g is linear;
- (iii). A_g is $\langle g \rangle$ -invariant;
- (iv). A_g is a closed subtree;
- (v). A_g is not properly contained in a linear subtree of X ;
- (vi). If T is a maximal linear g -invariant subtree of X , then $L = A_g$;
- (vii). If $x \in X$ then $[x, u] = \overline{\text{Br}}(x, A_g)$;
- (viii). $[x, gx] \cap A_g = [u, gu]$;
- (ix). $[x, gx] = [x, u, gu, gx]$;
- (x). $A_g = A_{g^n}$ for $n \neq 0$;
- (xi). $\mathbf{a}_x(g) = d(x, u)$;
- (xii). $\mathbf{b}_x(g) = d(u, gu)$.

Proof: Let $x \in X$, and let $u = u_x$ be as in the Theorem. Now $u \in [g^{-1}x, x]$ implies $gu \in [x, gx]$. Also $u \in [x, gx]$, so we have either $[x, u, gu, gx]$ or $[x, gu, u, gx]$. But in the latter case we have $u \in [gu, gx]$ and $gu \in [u, x]$, forcing $u, g^2u \in [gu, gx]$. Thus either $[gu, u, g^2u]$ holds, giving $[u, g^{-1}u, gu]$ or $[gu, g^2u, u]$ holds, giving $[u, gu, g^{-1}u]$. Both of these cases force g to be either a nesting reflection, an inversion or elliptic, contradicting our original assumption.

Therefore $[x, u, gu, gx]$ holds, giving part (9). We also have $[g^{-1}x, g^{-1}u, u, x]$. Now $u \in [g^{-1}x, gx]$ implies $[g^{-1}x, g^{-1}u, u, gx]$, whence $[g^{-1}x, g^{-1}u, u, gu, gx]$ and $[g^{-1}u, u, gu]$. Therefore $u \in A_g$. This shows that $A_g \neq \emptyset$, establishing part (1).

Next, for $p \in A_g$ put $A_p = \cup_{n \in \mathbb{Z}} [g^n p, g^{n+1} p]$. Then, since $[g^n p, g^{n+1} p, g^{n+2} p]$ holds for $n \in \mathbb{Z}$, the Piecewise Geodesic Proposition (see [10, 2.1.5]) and an induction argument give $[g^n p, g^m p, g^l p]$ for $n \leq m \leq l$, whence A_p is linear.

Now if $q \in A_p$, we have $[g^n p, q, g^{n+1} p]$ for some n , whence $[g^{n-1} p, g^{-1} q, g^n p]$ and $[g^{n+1} p, gq, g^{n+2} p]$, which gives $[g^{n-1} p, g^{-1} q, g^n p, q, g^{n+1} p, gq, g^{n+2} p]$, and hence $[g^{-1} q, q, gq]$. This gives $q \in A_g$. Therefore $A_p \subseteq A_g$, and consequently $\cup_{p \in A_g} A_p \subseteq A_g$. It is now easy to see that A_g is $\langle g \rangle$ -invariant, since each A_p clearly is. This proves (3).

For $p \in A_g$, put $A_p^+ = \cup_{n \geq 0} [p, g^n p]$, and $A_p^- = \cup_{n \leq 0} [p, g^n p]$. Then $A_p = A_p^+ \cup A_p^-$ and $A_p^+ \cap A_p^- = \{p\}$. We observe that if J is a subtree of A_p containing p then $gJ \subseteq J$ implies $A_p^+ \subseteq J$, and $J \subseteq gJ$ implies $A_p^- \subseteq J$.

Claim 1. A_p is a minimal (non-empty) $\langle g \rangle$ -invariant subtree of X . Moreover, for $p, q \in A_g$, either $A_p = A_q$ or $A_p \cap A_q = \emptyset$.

To see this, suppose that J is a non-empty $\langle g \rangle$ -invariant subtree of A_p . If $x \in J$ then $x \in [g^m p, g^{m+1} p]$ for some m . Thus $g^{-1}x \in [g^{m-1} p, g^m p]$, giving $[g^{m-1} p, g^{-1}x, g^m p, x, g^{m+1} p]$, whence $g^m p \in [g^{-1}x, x] \subseteq J$. Since J is $\langle g \rangle$ -invariant, this gives $p \in J$. Applying the observation above to J , we obtain $A_p^+ \subseteq J$ and $A_p^- \subseteq J$, that is, $A_p \subseteq J$. This forces $J = A_p$.

For the second assertion, put $J = A_p \cap A_q$. If J is non-empty, then it is a $\langle g \rangle$ -invariant subtree of A_p , which now implies $J = A_p$, whence $A_p = A_q$.

Claim 2. $A_p \cup A_q$ is linear for $p, q \in A_g$.

Suppose that $A_p \cup A_q$ is not linear. Then A_p and A_q are disjoint and linear, so swapping p and q if necessary, we see that there exists $p_0 \in A_p$ with $[p_0, q'] \cap A_p = \{p_0\}$ for all $q' \in A_q$, by Lemma 1.10. Moreover $p_0 \in [p', q']$ for all $p' \in A_p$ and $q' \in A_q$.

But $[gp_0, gq']$ is a segment joining a point of A_p to a point of A_q , forcing $p_0 \in [gp_0, gq'] \cap A_p = \{gp_0\}$. Thus $gp_0 = p_0$, a contradiction.

Claim 3. Any three points of A_g are collinear.

Choose $p, q, r \in A_g$. If A_p, A_q and A_r are not pairwise disjoint, then two of these subtrees coincide, and this claim reduces to the previous claim. So suppose that they are pairwise disjoint. If p, q and r are not collinear, let $w = Y(p, q, r)$. Then $w \in [p, q]$. If $w \in A_p$, then $w \in [q, r]$, giving $g^n w \in [g^n q, g^n r]$, a segment with endpoints belonging to A_q and A_r . But the subtree I spanned by $A_q \cup A_r$ is linear, with A_q and A_r as $\langle g \rangle$ -invariant subtrees, so this forces $A_p = A_w = \cup_{n \in \mathbb{Z}} [g^n w, g^{n+1} w] \subseteq I$, whence A_p, A_q and A_r are collinear, a contradiction.

So we may suppose that $w \notin A_p$, and likewise that $w \notin A_q \cup A_r$. Then $w \in \text{Br}^o(A_p, A_q)$ and similarly $w \in \text{Br}^o(A_p, A_r) \cap \text{Br}^o(A_q, A_r)$, so that w is the unique point common to all three open bridges. By g -invariance of A_p, A_q and A_r , we also see that gw is the unique point common to the three open bridges, forcing $gw = w$, a contradiction. This proves part (2).

Claim 4. A_g is a subtree.

Let $p, q \in A_g$; it suffices to show that $[p, q] \subseteq A_g$. If $A_p \cap A_q \neq \emptyset$, then $A_p = A_q$. It is clear in this case that $[p, q] \subseteq A_p \subseteq A_g$, since A_p is a subtree.

So suppose that $A_p \cap A_q = \emptyset$, and let $r \in \text{Br}^o(A_p, A_q)$. Then $gr \in \text{Br}^o(gA_p, gA_q) = \text{Br}^o(A_p, A_q)$, and likewise $g^{-1}r \in \text{Br}^o(A_p, A_q)$. Since the bridge between disjoint subtrees is linear, the points $g^{-1}r, r$ and gr are collinear. Since g is assumed not to be a nesting reflection, this forces $[g^{-1}r, r, gr]$. That is, $r \in A_g$. This shows that $\text{Br}^o(A_p, A_q) \subseteq A_g$. Now $[p, q] \subseteq A_p \cup \text{Br}^o(A_p, A_q) \cup A_q \subseteq A_g$.

Claim 5. A_g is a maximal linear subtree of X .

Suppose that A_g is properly contained in a linear subtree L of X with $x \in L \setminus A_g$. We have shown above that $u = Y(g^{-1}, x, gx) \in A_g$, and consequently $gu, g^{-1}u \in A_g$. Swapping the roles of g and g^{-1} if necessary, we now have $[g^{-1}u, u, gu, x]$, since A_g is a subtree and $x \notin A_g$. But in general $[x, gx] = [x, u; gu, gx]$, contradicting the configuration just stated. Therefore A_g is not properly contained in a linear subtree. This shows that (5) is satisfied. Moreover, since $A_g \subseteq A_{g^n}$ and g^n is hyperbolic for $n \neq 0$ by Lemma 1.7, part (10) follows.

Claim 6. A_g is a closed subtree of X .

We have seen that $[x, gx] = [x, u_x, gu_x, gx]$, that $u_x, gu_x \in A_g$ and that A_g is a subtree. If $y \in [x, u_x]$ then $[x, y, u_x, gu_x, gy, gx]$, whence $u_x \in [y, gy]$. Also, $gu_x \in [y, gy]$ giving $u_x \in [g^{-1}y, y]$, and $[g^{-1}x, g^{-1}u_x, u_x, gu_x, gx]$, giving $u_x \in [g^{-1}y, gy]$. So $u_x = Y(g^{-1}y, y, gy) = u_y$, giving $[g^{-1}y, y] \cap [y, gy] = [y, u_x]$. Therefore $y \in A_g$ if and only if $y = u_x$, giving $[x, u_x] \cap A_g = \{u_x\}$, whence $[gx, gu_x] \cap A_g = \{gu_x\}$ and $[x, gx] \cap A_g = [u_x, gu_x]$. This proves (8).

For arbitrary $x, y \in X$ with $[x, y] \cap A_g \neq \emptyset$, let $q \in [x, y] \cap A_g$. Then $[x, q] = [x, u_x, q]$ and $[q, y] = [q, u_y, y]$, giving $[x, y] = [x, u_x, q, u_y, y]$, and thus $[x, y] \cap A_g = [u_x, u_y]$. Therefore A_g is a closed subtree, as claimed in (4).

Writing $u_x = u(x, g)$, we have also shown that $[x, u(x, g)] = \overline{\text{Br}}(x, A_g)$ where $u(x, g) = Y(g^{-1}x, x, gx)$, which establishes (7). Thus $d(x, A_g) = d(x, u(x, g))$.

Now suppose that T is a maximal linear subtree which is g -invariant, and that $x \in T \setminus A_g$. Since T is g -invariant, we have $g^{-1}x, x, gx \in T$. Since A_g is also g -invariant, these points do not belong to A_g . But $u = Y(g^{-1}x, x, gx) \in \{g^{-1}x, x, gx\}$ since these points belong to T which is linear, and $u \in A_g$, a contradiction. Therefore $T \subseteq A_g$, whence (6).

All the assertions have now been established with the exception of the last two. Note that since $A_g = A_{g^n}$ for $n \neq 0$, we obtain $u(x, g^n) = u(x, g)$ for $n \neq 0$. It follows that $L(g^2) = d(x, u) + d(u, gu) + d(gu, g^2u) + d(g^2u, g^2x)$, and $L(g^{-1}) = d(x, u) + d(u, g^{-1}u) + d(g^{-1}u, g^{-1}x)$. Expanding the expression for $\mathbf{b}_x(g)$, it is now straightforward to show that it is equal to $d(u, gu)$, and similarly that $\mathbf{a}_x(g) = d(x, u)$. \square

This theorem shows that in many respects affine hyperbolic automorphisms behave like their isometric counterparts.

We next give a partial characterisation of the different types of affine automorphism of a Λ -tree in terms of based length functions, via the ancillary function $\mathbf{b} = \mathbf{b}_x$. In fact \mathbf{b} plays a role similar to the hyperbolic length function ℓ in the isometric case. If g is an isometry and not an inversion then $\mathbf{b}(g) = \ell(g)$. In particular \mathbf{b} is independent of the basepoint in this case; there is however no way of escaping the dependence of \mathbf{b} on the choice of basepoint in general.

Proposition 1.12. *Let g be an affine automorphism of a Λ -tree X . Then exactly one of the following holds.*

- (i). $\emptyset \neq X^g$ — that is, g is elliptic.
- (ii). $\emptyset = X^g = A_g \subset \tilde{A}_g$, and g is a nesting reflection or an inversion. Moreover $\mathbf{b}_x(g) < 0$ for all x in this case.
- (iii). $\emptyset = X^g \subset A_g = \tilde{A}_g$, and g is hyperbolic. Moreover $\mathbf{b}_x(g) > 0$ for all x in this case.

Proof: It is clear from the respective set inclusions that the cases are mutually exclusive, and that $X^g \neq \emptyset$ precisely when g is elliptic. Moreover $A_g = \tilde{A}_g$ if g is hyperbolic by Lemma 1.7(2) and proper inclusion holds if g is a nesting reflection or an inversion.

Fix $x \in X$, and put $u = Y(g^{-1}x, x, gx)$. It is clear from Theorem 1.11(12) that $\mathbf{b}_x(g) = d(u, gu) > 0$ for all x if g is hyperbolic. So suppose that g is a nesting reflection or an inversion. In this case $gu, g^{-1}u \in [x, u]$, for otherwise (swapping g and g^{-1} if necessary) $gu \in [u, gx]$ which implies that $[g^{-1}u; u; gu]$ which is impossible for a nesting reflection or an inversion, by Lemma 1.7(4). If $[x, g^{-1}u, gu, u]$ then one can show that the following configuration holds: $[g^{-1}x, g^{-2}u, u, g^2u, gu, g^{-1}u, x]$, and hence $[g^{-1}x, u, gu, g^{-1}u, x]$. Using this and noting

that $gu = Y(x, gx, g^2x)$, the defining expression for $\mathbf{b}_x(g)$ simplifies to $-d(u, gu)$. If instead $[x, gu, g^{-1}u, u]$, then the foregoing shows that $\mathbf{b}_x(g^{-1}) = -d(u, g^{-1}u)$. It can be shown directly from the definition of \mathbf{b}_x and using (L3) that $\mathbf{b}_x(g) = \alpha_g \mathbf{b}_x(g^{-1}) = -\alpha_g d(u, g^{-1}u) = -d(u, gu)$, as required. \square

Of course we can further distinguish between nesting reflections and inversions: in the former case $X^{g^2} = \emptyset$ and $\mathbf{b}_x(g^2) > 0$ for all x , while in the latter case $X^{g^2} \neq \emptyset$ and $\mathbf{b}_x(g^2) = 0$ for some x .

On the other hand, elliptic automorphisms cannot be distinguished from other types of automorphism by considering \mathbf{b} and A_g and \tilde{A}_g alone.

Recall that if g is a nesting reflection then g^2 is hyperbolic. We thus have a useful criterion for recognising free actions.

Corollary 1.13. (i). *An affine action of a group on a Λ -tree is free and without inversions if and only if g^2 is hyperbolic for all $g \neq 1$;*

(ii). *If an affine action is free (possibly with inversions) then g^2 is hyperbolic for all $g^2 \neq 1$.*

(iii). *If an affine action is free, rigid and without inversions, then g is hyperbolic for all $g \neq 1$.* \square

To show that a free product of ATF groups is again ATF, we will embed a family of ordered abelian groups in (a subgroup Λ of) their Cartesian product and then use the base change functor to obtain actions of the given groups on Λ -trees. The next step is to show that freeness is preserved by the application of the base change functor. However it is not true in general that the base change functor preserves freeness, even if the map h is an embedding — Example 1.9 shows how it may fail (take $\Lambda_1 = \Lambda_0$ and $\Lambda_2 = \mathbb{R}$, and take h to be the inclusion). However, in the case of interest to us this argument is valid. To show this, we need a somewhat technical result which may be of independent use.

Fix an α -affine action of G on a Λ -tree X , a basepoint $x \in X$ and $g \in G$, and suppose that $[g^{-1}u; u; gu]$ where as usual $u = Y(g^{-1}x; x; gx)$ (so g is hyperbolic on the subtree spanned by $\langle g \rangle x$). Note that $u \in A_g$. We define the *right radius of A_g with respect to x* to be

$$\text{Rad}^r(g) = \text{Rad}_x^r(g) = \{d(u, v) : v \in A_g, u \notin [gu, v]\} \cup \{0\}$$

The *left radius of A_g with respect to x* , $\text{Rad}^l(g)$, is the right radius of $A_{g^{-1}}$ with respect to x . Of course this amounts to an abuse of notation, since these sets depend not just on the set $A_g = A_{g^{-1}}$, but on g . However it is convenient in this context to think of A_g as having an orientation determined by g and we will do so. (Our terminology implies that g translates ‘to the right’, at least on the subtree spanned by $\langle g \rangle u$.)

The *diameter of A_g with respect to x* is $\text{Diam}(g) = (-\text{Rad}^l(g)) \cup (\text{Rad}^r(g))$. Clearly $\text{Diam}(g^{-1}) = -\text{Diam}(g)$. Moreover, $\text{Diam}(g)$ is naturally isometric to A_g , via a map which sends 0 to u .

In Example 1.9, if we take the subtree $X_0 = (-\infty, 4/3]$ of \mathbb{R} , then X_0 is invariant under g^2 . Taking $x = u = 1$, we see that $\text{Rad}^r(g^2) = [0, \infty)$, and $\text{Rad}^l(g^2) = [0, 1/3]$, so that $\text{Diam}(g) = [-1/3, \infty)$. If instead we define X_0 to exclude the fixed point $4/3$ then $\text{Rad}^l(g) = [0, 1/3)$, so that $\text{Diam}(g) = (-1/3, \infty)$.

Proposition 1.14. *Suppose that X is spanned by the orbit Gx and that $\mathbf{b}(g) > 0$ (where $\mathbf{b} = \mathbf{b}_x$). Then g has a fixed point if and only if $(1 - \alpha_g)^{-1}\mathbf{b}(g) \cap \text{Rad}^r(g) \neq \emptyset$ or $(1 - \alpha_{g^{-1}})^{-1}\mathbf{b}(g^{-1}) \cap \text{Rad}^r(g^{-1}) \neq \emptyset$.*

Proof: Note first that $\mathbf{b}_x(g) = d(u, gu)$ if $\mathbf{b}_x(g) > 0$; this can be seen for example by applying Theorem 1.11(12) to the subtree of X spanned by $\langle g \rangle x$.

Suppose that y is fixed by g . Then $y \in A_g$ and, since $\mathbf{b}(g) > 0$, y cannot lie in the subtree T spanned by $\langle g \rangle u$. Replacing g by its inverse if necessary we have $[u, gu, y]$, whence

$$\begin{aligned} \alpha_g d(u, y) &= d(gu, gy) \\ &= d(gu, y) \\ &= d(u, y) - d(u, gu) \\ &= d(u, y) - \mathbf{b}(g), \end{aligned}$$

forcing $d(u, y) \in (1 - \alpha_g)^{-1} \mathbf{b}(g)$ and clearly $d(u, y) \in \text{Rad}^r(g)$.

Conversely, we have $[g^{-1}u; u; gu]$, so that $\cup_{n \in \mathbb{Z}} [g^{n-1}u, g^nu]$ is a linear $\langle g \rangle$ -invariant subtree on which g has no fixed point. Replacing g by its inverse if necessary, our hypothesis guarantees the existence of λ such that $(1 - \alpha_g)(\lambda) = \mathbf{b}(g)$ and λ lies in $\text{Rad}^r(g)$. Since, moreover, X is spanned by Gx , $[g^{-1}u; u, y, \gamma x]$ for some $\gamma \in G$. We claim that $T \cup \{g^{-1}y, y, gy\}$ is a collinear set. Since T is $\langle g \rangle$ -invariant, it suffices to show that $T \cup \{g^{-1}y, y\}$ or $T \cup \{y, gy\}$ is collinear. If $y \in T$, the claim is obvious, and if y is in the subtree spanned by $\langle g \rangle x$ but $y \notin T$ the configuration $[g^{-1}y, y, gy]$ is impossible. So suppose otherwise. Since we have $[g^{\pm 1}t; t, y]$ for all $t \in T$, there can be no point $x_1 \in T$ such that $x_1 \in [t, y]$ for all $t \in T$. Therefore, applying Lemma 1.10 to $I_1 = [g^{-1}y, y, gy]$ and $I_2 = T$, there exists $x_0 \in [g^{-1}y, y, gy]$ such that $[t; x_0, g^{-1}y]$ and $[t; x_0, y]$ for all $t \in T$. Thus either $\{t; x_0, g^{-1}y, y\}$ is collinear for all $t \in T$ $x_0 \in [g^{-1}y, y]$ so that $[g^{-1}y, x_0, y, gy]$ and $[t, x_0, y, gy]$ is collinear for all $t \in T$. The claim follows.

We therefore have either $[u, y, gy]$ or $[u, gy, y]$, the configuration $[y, u, gy]$ being forbidden by our choice of y . Now $gu \in [u, gy]$ and $\lambda = d(u, y)$ imply

$$\begin{aligned} \alpha_g(\lambda) &= d(gu, gy) \\ &= d(u, gy) - d(u, gu). \end{aligned}$$

Therefore our choice of λ gives

$$\begin{aligned} \lambda = \mathbf{b}(g) + \alpha_g(\lambda) &= d(u, gy) \\ &= d(u, y) \pm d(y, gy) \\ &= \lambda \pm d(y, gy) \end{aligned}$$

This forces $y = gy$. □

The latter sentence of Proposition 1.14 may be restated as follows: g fixes a point if and only if $(1 - \alpha_g)^{-1} \mathbf{b}(g) \cap \text{Diam}(g) \neq \emptyset$.

We conclude this discussion by observing that if $\mathbf{b}_x(g') > 0$ for all conjugates g' of g then $\text{Rad}^r(g)$ (and hence $\text{Diam}(g)$) can be characterised in terms of based length functions $L = L_x$ in the case where X is spanned by Gx . Note first that $\mathbf{b}_{\gamma x}(g) = \alpha_\gamma \mathbf{b}_x(\gamma^{-1}g\gamma)$ for $\gamma \in G$, so that $\mathbf{b}_{\gamma x}(g) > 0$ for all γ . So in the notation of Theorem 1.11 if $w = u(\gamma x, g)$ then we have $[g^{-1}w; w; gw]$. It follows that $\alpha_\gamma \mathbf{a}_x(\gamma^{-1}g\gamma) = \mathbf{a}_{\gamma x}(g) = d(\gamma x, w) = d(\gamma x, A_g)$, so that $\text{Rad}^r(g)$ is spanned by elements of the form $d(u, w)$ where $[x, u, w, \gamma x]$ and gu and γx represent the same direction at u — in other words $\mathbf{c}_x(g, \gamma) > a_x(g)$. Thus

Proposition 1.15. *$\text{Rad}^r(g)$ is spanned by elements of the form*

$$L_x(\gamma) - \mathbf{a}_x(g) - \alpha_\gamma \mathbf{a}_x(\gamma^{-1}g\gamma) \text{ where } \mathbf{c}_x(g, \gamma) > a_x(g).$$

□

1.4 Line stabilisers

A *line* in a Λ -tree is a maximal linear subtree. Let G be a group that admits a free affine action (without inversions) on a Λ -tree X . We will call a subgroup H a *line stabiliser* for the action of G if there is a line T of X whose stabiliser in G is H .

Recall that the full group of isometric automorphisms of Λ considered as a Λ -tree is isomorphic to $\Lambda \rtimes C_2$: here Λ acts on itself by translation, and the non-trivial element of C_2 changes the sign (see [1, 2.5]). It is not hard to generalise this to the affine case: the group of affine automorphisms of Λ itself is

$$\text{Aff}(\Lambda) \cong \Lambda \rtimes (\text{Aut}^+(\Lambda) \times C_2).$$

(It is worth noting first that the set of permutations g of a Λ -tree X that are α_g -affine automorphisms for some $\alpha_g \in \text{Aut}^+(\Lambda)$ forms a group.)

Let Λ be an ordered abelian group of finite rank. As noted in §1.2, if Λ_0 is a convex subgroup, order-preserving automorphisms of Λ_0 extend to Λ , and $\text{Aut}^+(\Lambda_0)$ embeds in $\text{Aut}^+(\Lambda)$. Let T be

a linear Λ -tree, which we assume to be isometrically embedded in Λ , let $\alpha_g \in \text{Aut}^+(\Lambda_0)$ where Λ_0 is spanned by $\{d(x, y) : x, y \in T\}$, and g an affine automorphism of T with dilation factor α_g . Let $\bar{\alpha}_g \in \text{Aut}^+(\Lambda)$ be an extension of α_g . We claim that g extends to an $\bar{\alpha}_g$ -affine automorphism Λ . First we can define an isometry $\sigma_g : T \rightarrow \Lambda$ via $x \mapsto (\bar{\alpha}_g)^{-1}gx$. It is easy to see that σ_g is isometric, and thus by [10, Lemma 2.3.1] σ_g extends to an isometric automorphism $\bar{\sigma}_g$ of Λ . It follows that $\bar{\alpha}_g\bar{\sigma}_g$ is an $\bar{\alpha}_g$ -affine automorphism of Λ ; it is routine to check that $\bar{\alpha}_g\bar{\sigma}_g$ extends the action of g on T .

We deduce that an affine action of a group on a linear subtree of Λ extends to an affine action on Λ provided Λ has finite rank. (In fact the assumption of finite rank is only needed to ensure that the dilation factors extend to Λ .)

The following observation will also be useful. Let G be a group acting on a set X , N a normal subgroup, and let $X_0 = \{x \in X : gx = x \ \forall g \in N\}$. Then G stabilises X_0 .

Proposition 1.16. *Let G be a group that admits a free affine action on a Λ -tree X , and H a subgroup of G .*

- (i). *If H is a non-trivial subnormal abelian subgroup of G then G stabilises a line.*
- (ii). *Suppose that $\text{Aut}^+(\Lambda)$ is soluble and H stabilises a line of X . Then H is soluble.*
- (iii). *Suppose that $\Lambda = \mathbb{Z}^n$ for some n and H stabilises a line T of X and preserves the orientation. Then H is nilpotent.*

Proof: (1) For $g \neq 1$ we have g^2 hyperbolic by Corollary 1.13(1). Thus g^2 stabilises a line (A_{g^2}) and preserves the orientation of the line. If ϵ and ϵ' are the ends of this line then $g^2\epsilon = \epsilon$ and $g^2\epsilon' = \epsilon'$, whence g stabilises $\{\epsilon, \epsilon'\}$. Thus g stabilises a line, which must be unique since otherwise g would fix a branch point.

If $gh = hg$ where g and h are non-trivial, then g and h must stabilise a common line. It follows that a non-trivial abelian subgroup A of G stabilises a unique line. The result follows by the observation above and an easy induction on n .

(2) The subgroup H acts faithfully on any line that it stabilises, whence H embeds in $\Lambda \rtimes (\text{Aut}^+(\Lambda) \times C_2)$. The result follows since Λ is abelian and $\text{Aut}^+(\Lambda)$ is soluble.

(3) The full group of orientation-preserving affine automorphisms of T embeds in that of \mathbb{Z}^n , which is isomorphic to $\mathbb{Z}^n \rtimes \text{Aut}^+(\mathbb{Z}^n) \cong \text{UT}(n+1, \mathbb{Z})$, which is nilpotent. Thus, since H acts faithfully on T , H is nilpotent. \square

Corollary 1.17. *Suppose that G is non-trivial and has a free affine action on a Λ -tree.*

- (i). *If $\text{Aut}^+(\Lambda)$ is soluble, the non-trivial line stabilisers are precisely the maximal soluble subgroups of G .*
- (ii). *If $\Lambda = \mathbb{Z}^n$ and there is no line whose ends are interchanged by any $g \in G$, then the non-trivial line stabilisers are precisely the maximal nilpotent subgroups of G .*

Proof: (1) If $H \neq 1$ stabilises a line, then H is soluble by Proposition 1.16. If $\langle H, \gamma \rangle$ is soluble, then by Proposition 1.16 this subgroup stabilises a line, which must be the same line as that stabilised by H . It follows that if H is a line stabiliser then H is maximal soluble.

Conversely, if H is non-trivial and soluble, H normalises the last non-trivial term of its derived series, which is a non-trivial abelian group, so that H stabilises a line, by Proposition 1.16. If γ also stabilises this line, then $\langle H, \gamma \rangle$ is soluble. Therefore if H is maximal soluble, then H is a line stabiliser.

An analogous argument establishes (2). \square

Note that the class of Λ for which $\text{Aut}^+(\Lambda)$ is soluble includes all ordered abelian groups of finite rank: this follows from Corollary 1.2.

Recall that in the isometric case, if a group acts freely on a Λ -tree X , and an end is fixed, then the group acts by translations on an invariant subtree. Thus each end stabiliser is a line stabiliser in the isometric case. We next show that the same is true of affine actions on Λ -trees provided $\text{Aut}^+(\Lambda)$ is soluble. (In fact, the statement is true under the slightly weaker hypothesis that the image of α is soluble.)

Proposition 1.18. *Let Λ be an ordered abelian group for which $\text{Aut}^+(\Lambda)$ is soluble. Let G be a group that has a free affine action on a Λ -tree X , and suppose that G fixes an end ϵ of X . There is a unique line (ϵ, ϵ') stabilised by G .*

Proof: Use induction on the derived length $d = d_G$ of $\alpha(G) \leq \text{Aut}^+(\Lambda)$. If $d = 0$ then $G = \ker \alpha$ and the given action is isometric. The proposition is true in the isometric case, as we have already observed. Now if $[G, G] = 1$ then G is abelian, and by Proposition 1.16 G stabilises a unique line.

Assume that the result true for groups G_1 with $d_{G_1} < d$, and that $d \geq 2$. Then $[G, G]$ stabilises a unique line of X . Thus $[G, G]$ fixes exactly two ends, one of which must be ϵ , and the other which we denote by ϵ' . Therefore G stabilises this pair of ends; since G fixes ϵ , it must also fix ϵ' , whence G stabilises the line (ϵ, ϵ') . Since $[G, G]$ fixes no other line, neither does G . \square

Recall that a CSA group is one in which all maximal abelian subgroups are malnormal. This means that for CSA groups G , and maximal abelian subgroups M , if $M \cap M^\gamma \neq 1$, then $\gamma \in M$. Note that this implies that each non-trivial element of G is contained in a unique maximal abelian subgroup — equivalently G is commutative-transitive (commutativity is an equivalence relation on its non-trivial elements).

Let us consider these properties in a more general setting. Let \mathcal{P} be a class of groups that is subgroup-closed. Say that a group G is \mathcal{P} -disconnected if each non-trivial $x \in G$ is contained in a unique maximal \mathcal{P} subgroup of G , called *the \mathcal{P} -component of x* . (So abelian-disconnected groups are precisely commutative-transitive groups.) Call a group G CSP if G is \mathcal{P} -disconnected and the maximal \mathcal{P} subgroups of G are malnormal.

Some cases of interest are where \mathcal{P} is A , the class of abelian groups, S , the class of soluble groups, and N , the class of nilpotent groups.

Proposition 1.19. *(i). Let Λ be an ordered abelian group with $\text{Aut}^+(\Lambda)$ soluble. Then an $\text{ATF}(\Lambda)$ group is CSS .*

(ii). If G admits a free affine action on a \mathbb{Z}^n -tree X and no line of X has its ends interchanged by an element of G then G is CSN .

Proof: (1) and (2) may be proven by taking $\mathcal{P} = S$ and $\mathcal{P} = N$, and invoking Corollary 1.17(1) and (2) respectively. Let M be a maximal \mathcal{P} subgroup of a group G equipped with a free α -affine action on a Λ -tree. There exists a unique line $L = (\epsilon, \epsilon')$ in X whose stabiliser is M by Corollary 1.17. Further, the subgroup $\gamma M \gamma^{-1}$ is also maximal \mathcal{P} and stabilises γL . If h belongs to the intersection, then h stabilises L and γL ; but this forces $L = \gamma L$ since h stabilises a unique line. Thus γ belongs to the stabiliser of L , namely M . \square

We call a group G an $\text{ATF}[\mathcal{P}]$ group if it admits a free affine action on a Λ -tree X for some Λ such that all line stabilisers are \mathcal{P} subgroups, and every \mathcal{P} subgroup of G stabilises a line (which must be unique). The argument of Proposition 1.19 shows that $\text{ATF}[\mathcal{P}]$ groups are CSP (and hence \mathcal{P} -disconnected). The following proposition follows the idea of [10, Theorem 5.5.3] (which is largely due to B. Baumslag and Remeslennikov) in the case where \mathcal{P} is the variety of abelian groups, and free groups are considered in place of $\text{ATF}[\mathcal{P}]$ groups. In this case the assumption that G is not a \mathcal{P} group can be dropped.

The following proposition is in fact true if $\text{ATF}[\mathcal{P}]$ is replaced by any group theoretic property stronger than CSP .

Proposition 1.20. *Let \mathcal{P} be a variety of groups that contains all abelian groups, and suppose that G is residually $\text{ATF}[\mathcal{P}]$, but that G is not a \mathcal{P} group. The following are equivalent.*

- (i). G is fully residually $\text{ATF}[\mathcal{P}]$.*
- (ii). G is 2-residually $\text{ATF}[\mathcal{P}]$.*
- (iii). G is \mathcal{P} -disconnected.*
- (iv). G is CSP .*

Proof: (1) \Rightarrow (2) is trivial. Suppose that (2) holds. If H is a \mathcal{P} subgroup of G then any homomorphism $G \rightarrow Q$ maps H to a \mathcal{P} subgroup of Q . Let $x, y, z \in G$ with $y \neq 1$ and suppose that $\langle x, y \rangle$ and $\langle y, z \rangle$ are \mathcal{P} subgroups: we claim that $H = \langle x, y, z \rangle$ is \mathcal{P} . For taking any law $w = w(x_1, \dots, x_r)$ of \mathcal{P} , let $g = w(g_1, \dots, g_r)$ where $g_i \in H$ for all i . If $g \neq 1$ we can apply (2) to obtain a homomorphism $\phi : G \rightarrow Q$ where Q is $\text{ATF}[\mathcal{P}]$, and hence \mathcal{P} -disconnected, and where $\phi(g) \neq 1 \neq \phi(y)$. Now $\phi(x)$ and $\phi(z)$ both belong to the maximal \mathcal{P} subgroup containing $\phi(y)$. Hence $\phi(g) = w(\phi(g_1), \dots, \phi(g_r)) = 1$, a contradiction. Thus $g = 1$ from which we conclude that H is a \mathcal{P} subgroup.

Next let $M = M_y = \{x \in G : \langle x, y \rangle \text{ is a } \mathcal{P} \text{ subgroup}\}$. It is now straightforward to show that M is in fact a subgroup. Moreover, every pair of (non-trivial) elements of M is mapped to the same \mathcal{P} component by any homomorphism with \mathcal{P} -disconnected codomain, which forces M to be a \mathcal{P} subgroup, using the 2-residual \mathcal{P} -disconnectedness of G . Clearly $\langle M, \gamma \rangle$ can only be a \mathcal{P} subgroup if $\gamma \in M$, and any \mathcal{P} subgroup of G containing y is contained in M ; therefore M is the unique maximal \mathcal{P} subgroup of G containing y . This proves (3).

Assume (3). Let M be a maximal \mathcal{P} subgroup of G , and suppose that $\gamma \notin M$. Then $\langle M, \gamma \rangle$ is not a \mathcal{P} subgroup and there exists a law w for the variety \mathcal{P} such that $g = w(g_1, \dots, g_r) \neq 1$ for some choice of $g_i \in \langle M, g \rangle$ ($1 \leq i \leq r$). Let Q be an $\text{ATF}[\mathcal{P}]$ group and $\phi : G \rightarrow Q$ a homomorphism with $\phi(g) \neq 1$. Now $\phi(M)$ is a \mathcal{P} subgroup of Q which is thus contained in a unique maximal \mathcal{P} subgroup \bar{M} , since $\text{ATF}[\mathcal{P}]$ groups are \mathcal{P} -disconnected. If $\phi(\gamma) \in \bar{M}$, then $\langle \phi(M), \phi(\gamma) \rangle$ is a \mathcal{P} subgroup of Q . But then $\phi(g) = w(\phi(g_1), \dots, \phi(g_r))$ with each $\phi(g_i)$ belonging to the \mathcal{P} subgroup $\langle \phi(M), \phi(\gamma) \rangle$. This forces $\phi(g) = 1$, a contradiction.

Therefore $\phi(\gamma) \notin \bar{M}$, giving $\bar{M} \cap \phi(\gamma)\bar{M}\phi(\gamma^{-1}) = 1$ since Q is CSP . Suppose that M and $\gamma M \gamma^{-1}$ have non-trivial intersection. Since these are maximal \mathcal{P} subgroups and G is \mathcal{P} -disconnected, they must coincide. But $\phi(M) \subseteq \bar{M}$ and $\phi(\gamma M \gamma^{-1}) \subseteq \phi(\gamma)\bar{M}\phi(\gamma^{-1})$, which gives a contradiction. Therefore $M \cap \gamma M \gamma^{-1} = 1$, whence (4).

Finally assume (4). Then for non-trivial $x \in G$, we can find $g \in G$ such that g does not belong to the \mathcal{P} component M of x . Thus $M \cap g M g^{-1} = 1$. Let y be another non-trivial element of G . If $[x, y] = 1$ then x and y must belong to the same \mathcal{P} component of G . We therefore have $g x g^{-1}$ and y in distinct \mathcal{P} components. In particular $[g x g^{-1}, y] \neq 1$.

Now let x_1, \dots, x_n be given non-trivial elements of G . Choosing a suitable conjugate y_i of x_i as above we have $[y_1, y_2, \dots, y_n] \neq 1$. Using the residual $\text{ATF}[\mathcal{P}]$ property we can map G to an $\text{ATF}[\mathcal{P}]$ group Q with the image of $[y_1, y_2, \dots, y_n]$ non-trivial. This forces the image of each y_i to be non-trivial, and with it, the image of each x_i . This proves (1). \square

2 Examples

Of course all isometric actions are affine — this corresponds to the case where the given homomorphism α is trivial. Consequently, $\text{ITF}(\Lambda)$ implies $\text{ATF}(\Lambda)$ as already noted.

The group $\text{Hol}^+(\Lambda) = \Lambda \rtimes \text{Aut}^+(\Lambda)$ acts naturally on Λ : this action is affine though not free in general.

Example 2.1 (Some free affine actions on \mathbb{R} -trees). In [20], I. Liousse constructs examples of groups that admit free affine actions on \mathbb{R} -trees. She constructs two types, the first having presentations of the form

$$\langle x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n \mid [x_1, y_1] = [x_2, y_2] = \dots = [x_n, y_n] \rangle,$$

and the second

$$\langle x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m \mid w = v^k \rangle$$

where w is a surface relator (that is, either $w = [x_1, x_2] \cdots [x_{2r-1}, x_{2r}]$ where $n = 2r$, or $w = x_1^2 \cdots x_n^2$) and v is a non-trivial word in the y_i^\pm .

As Liousse notes, it can be shown using Rips' Theorem that most of these groups do not admit free isometric actions on \mathbb{R} -trees. This shows, in our notation, that $\text{ATF}(\mathbb{R})$ does not imply $\text{ITF}(\mathbb{R})$.

Example 2.2 (A non-free affine action of a free group). Let F be the free group on $\{a_n : n \in \mathbb{Z}\}$, and Γ the corresponding Cayley graph. (So there is an edge labelled a_n joining γ to γa_n for $\gamma \in F$ and $n \in \mathbb{Z}$). Define the distance between two adjacent vertices (= length of the edge) to be 2^n if a_n is the label of the edge joining them. Extend to an \mathbb{R} -metric on the vertex set of Γ by putting $d(x, y)$ equal to the sum of the lengths of the edges in the reduced path joining x and y . This is clearly a 0-hyperbolic \mathbb{R} -metric.

Now left multiplication by F amounts to an isometric action on Γ . Moreover, we can define a ‘shifting’ map $\tau : F \rightarrow F$ by mapping an element $a_{i_1}^{\epsilon_1} a_{i_2}^{\epsilon_2} \cdots a_{i_m}^{\epsilon_m}$ to $a_{i_1+1}^{\epsilon_1} a_{i_2+1}^{\epsilon_2} \cdots a_{i_m+1}^{\epsilon_m}$. It is routine to verify that $d(\tau x, \tau y) = 2d(x, y)$ for $x, y \in F$, whence $G = \langle F, \tau \rangle$ has an affine action on Γ : the map $\alpha : G \rightarrow \text{Aut}^+(\mathbb{R})$ is given by $\gamma \mapsto 2^e$ where e is the exponent sum of τ in γ .

Now Γ embeds in an \mathbb{R} -tree in a natural way by Theorem 1.5. Moreover the action of G on Γ extends to an affine action on the \mathbb{R} -tree.

Observe next that $\tau^{-1} a_k \tau (a_{i_1}^{\epsilon_1} a_{i_2}^{\epsilon_2} \cdots a_{i_m}^{\epsilon_m}) = a_{k-1} (a_{i_1}^{\epsilon_1} a_{i_2}^{\epsilon_2} \cdots a_{i_m}^{\epsilon_m})$. Suppose that w is a word in τ and a_k ($k \in \mathbb{Z}$) which acts as the identity. Then $w \in \ker \alpha$, which means that the exponent sum of τ in w is zero. Therefore w is expressible as a product of terms of the form $\tau^{-k} a_l^{\pm 1} \tau^k$. Using the relation scheme noted above, w is in fact expressible as a product of $a_{l'}^{\pm 1}$ ($l' \in \mathbb{Z}$); that is, $w \in F$. This forces $w = 1$, since the action of F on its Cayley graph is free.

Therefore G has the presentation

$$\langle \tau, a_k \ (k \in \mathbb{Z}) \mid \tau^{-1} a_k \tau = a_{k-1} \rangle \cong \langle a_0, \tau \mid \rangle,$$

that is, G is the free group on $\{a_0, \tau\}$.

Example 2.3 (The Heisenberg group is ATF). Let $X = \Lambda = \mathbb{Z}^3$, and let $\sigma : (x, y, z) \mapsto (x, y+1, z)$, $\tau : (x, y, z) \mapsto (x+1, y, z+y)$. Let $G = \langle \sigma, \tau \rangle$. Then G has an α -affine action on X where $\alpha_\sigma = \text{id}$ and $\alpha_\tau(x, y, z) = (x, y, z+y)$.

Suppose that $w \in G$ maps $[r, s]$ into itself for some $r \leq s \in X$, so that (replacing w by w^2 if necessary) $r \leq wr \leq ws \leq s$. Then the exponent sum of τ in w must be zero so that w is a product of conjugates of the form $\tau^{-k} \sigma^l \tau^k$, each of which fixes the first entry of each element of X , and adds l to the second entry. Thus the exponent sum of σ in w is also zero, so that w lies in the derived subgroup of $\langle \sigma, \tau \rangle$. Direct computation shows that $[\sigma, [\sigma, \tau]] = [\tau, [\sigma, \tau]] = 1$; it follows that w is a product of commutators of the form $[\sigma^l, \tau^k]$. It is straightforward to show that $[\sigma^l, \tau^k](x, y, z) = (x, y, z - kl)$ and thus w is a product of commutators of the form $[\sigma^{l_i}, \tau^{k_i}]$ with $\sum k_i l_i = 0$, which therefore fixes X pointwise. This shows that the action is free and rigid.

The calculation above also implies

$$[\sigma, \tau]^{kl} = [\sigma^l, \tau^k], \quad k, l \in \mathbb{Z}.$$

It follows that

$$G \cong \langle \sigma, \tau \mid [\sigma, \tau]^{kl} = [\sigma^l, \tau^k], \ k, l \in \mathbb{Z} \rangle.$$

We claim that in fact G is isomorphic to $H = H_3(\mathbb{Z})$, the discrete Heisenberg group. It is well

known that H is generated by the matrices $x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, and that

$[x, [x, y]] = [y, [x, y]] = 1$ are defining relations.

Since $[\sigma, [\sigma, \tau]] = [\tau, [\sigma, \tau]] = 1$, there is an epimorphism $\phi : H \rightarrow G$ defined by $x \mapsto \sigma$ and $y \mapsto \tau$. Moreover, $[x^k, y^l] = [x, y]^{kl}$ for all $k, l \in \mathbb{Z}$, whence $\sigma \mapsto x$ and $\tau \mapsto y$ defines an epimorphism $\psi : G \rightarrow H$. Clearly $\phi \circ \psi = \text{id}$, whence both maps are isomorphisms.

Thus H admits a free rigid affine action on \mathbb{Z}^3 , viewed as a \mathbb{Z}^3 -tree, as claimed in Theorem 0.1(1).

Example 2.4 ($C_\infty \wr C_\infty$ and the soluble Baumslag-Solitar groups are ATF). Let $X = \Lambda = \mathbb{Z} \times \mathbb{R}$, and a a positive real number. Define $\sigma = \sigma_a : (x, y) \mapsto (x+1, ay)$ and $\tau : (x, y) \mapsto (x, y+1)$ and let $G_a = \langle \sigma_a, \tau \rangle$. Then $\sigma^{-k} \tau^l \sigma^k(x, y) = (x, y + \frac{l}{a^k})$. It follows that

$$[\sigma^{-k} \tau^l \sigma^k, \sigma^{-k'} \tau^{l'} \sigma^{k'}] = 1.$$

Let us write C for the set consisting of these relations (where k, k', l, l' range through the integers). Taking $H = \langle \tau \rangle$, and $K = \langle \sigma \rangle$ acting on the product $\prod_{n \in \mathbb{Z}} \sigma^{-n} H \sigma^n$ by conjugation, we see that G_a is a quotient of $H \wr K \cong \langle \sigma, \tau \mid C \rangle \cong C_\infty \wr C_\infty$.

Suppose that w is a word in σ and τ and that $w[r, s] \subseteq [r, s]$. Then the exponent sum of σ in w must be zero, so that w is a product of conjugates of τ by powers of σ . Since these conjugates commute as observed above we can write $w = \sigma^{-k_1} \tau^{l_1} \sigma^{k_1} \dots \sigma^{-k_r} \tau^{l_r} \sigma^{k_r}$ where $k_i < k_{i+1}$ for each i . Since the effect of such a w is to add $\sum_{i=1}^r l_i a^{-k_i}$ to the second entry, we must have $\sum_{i=1}^r l_i a^{-k_i} = 0$. This also shows that the action of G_a is free. Multiplying both sides of this equation by a^{k_r} , we obtain a polynomial equation. We next consider the consequences of this equation. Distinguish three cases.

- If a is transcendental, then all l_i are equal to zero since otherwise the polynomial equation above has a as a root. Therefore C amounts to a set of defining relations for G_a in this case, so that $G_a \cong C_\infty \wr C_\infty$.
- For any polynomial $p(x) \in \mathbb{Z}[x]$ for which $p(a) = 0$, we obtain a relation for the group G_a , as follows. Write $p(x) = \sum_{k=0}^n l_k x^k$ and put

$$w_p = \tau^{l_0} \cdot (\sigma \tau^{l_1} \sigma^{-1}) \dots (\sigma^n \tau^{l_n} \sigma^{-n}).$$

Then $w_p = 1$ is a relation of G_a . It is easily checked that if $p_1(x) = dx^m$, then $w_{p_1 p} = \sigma^m (w_p)^d \sigma^{-m}$, which is clearly equal to the identity in G_a if $w_p = 1$. Moreover, if $p_1(x), p_2(x) \in \mathbb{Z}[x]$ then $w_{(p_1+p_2)}$ is easily seen to be equal to $w_{p_1} w_{p_2}$. It follows that if $p_0(a) = 0$ and p_0 is a divisor of p , then w_p is a consequence of w_{p_0} and C .

Conversely, it is clear that if w is a word in σ and τ giving the identity in G_a , w must have the form w_p for some $p(x) \in \mathbb{Z}[x]$ with $p(a) = 0$. Therefore taking $p_0(x)$ to be the minimum polynomial of a , we obtain the following presentation.

$$G_a \cong \langle \sigma, \tau \mid w_{p_0} = 1, C \rangle$$

Of course if a_1 and a_2 have the same minimum polynomial, then G_{a_1} and G_{a_2} are isomorphic.

- If a is a (positive) integer, then

$$G_a \cong \langle \sigma, \tau \mid \sigma \tau \sigma^{-1} = \tau^a, C \rangle.$$

In fact the first relation implies that $\sigma^k \tau \sigma^{-k} = \tau^{a^k}$ and $\sigma^{k'} \tau \sigma^{-k'} = \tau^{a^{k'}}$ commute for $k, k' \geq 0$ — and hence for all $k, k' \in \mathbb{Z}$. It follows that

$$G_a \cong \langle \sigma, \tau \mid \sigma \tau \sigma^{-1} = \tau^a \rangle \cong \text{BS}(1, a),$$

a soluble Baumslag-Solitar group.

If a^{-1} is an integer, then replacing τ by its inverse we also find that G_a is isomorphic to $\text{BS}(1, a)$.

The group G_a for any transcendental a gives the example promised by Theorem 0.1(2), while taking $a \in \mathbb{N}$ gives the example referred to in Theorem 0.1(3). Moreover all actions under consideration here are in fact rigid.

3 Constructions

We say that a length function L on a group G is *regular* if for all $g, h \in G$, there exists $u \in G$ and $g', h' \in G$ with $L(u) = \mathbf{c}(g, h)$ where $L(g) = L(u) + \alpha_u L(g')$ and $L(h) = L(u) + \alpha_u L(h')$, and $g = ug', h = uh'$. In the isometric case, this boils down to the notion of regular length function in the sense of Myasnikov and Remeslennikov. Call an action regular if there exists a basepoint with respect to which the associated based length function is regular.

3.1 Free products

If γ is an element of a free product, we use the notation $\gamma = \gamma_1 \cdot \gamma_2 \cdots \gamma_k$ if the last syllable of each γ_i (when γ_i written as a reduced word in elements of the free factors) belongs to a different free factor from the first syllable of γ_{i+1} . (We do not assume that each γ_i belongs to a free factor.) Equivalently, the syllable length of γ as an element of the free product is equal to the sum of the syllable lengths of the γ_i .

Theorem 3.1. (see Theorem 0.3)

Let G_i be a group admitting an affine action on a Λ -tree for each $i \in I$. The free product $G = *_{i \in I} G_i$ has an affine action on a Λ -tree which extends the given actions of G_i .

If the given actions of G_i are respectively

- (i). free,
- (ii). free and rigid,
- (iii). or regular

then so is that of G .

Proof: Suppose that for each i we have an α_i -affine action on a Λ -tree. Take a basepoint x_i from each Λ -tree, and let $L = L_{x_i}$ denote the associated based length function.

Let $\alpha : G \rightarrow \text{Aut}^+(\Lambda)$ be the unique common extension of the α_i . If g_1, g_2, \dots, g_n are group elements, we define $\overline{g_k} = g_1 \cdots g_k$ for $1 \leq k \leq n$ and $g_0 = \overline{g_0} = 1$. It is also convenient to put $g_m = 1$ for $m > n$.

Now for $g = \overline{g_n}$ with consecutive g_i in distinct free factors, we put

$$L(g) = \sum_{k=1}^n \alpha_{\overline{g_{k-1}}} L(g_k).$$

(Here we abuse notation by using the same notation L for the length function on each G_i and for the length function on $G = *G_i$; similar abuses follow regarding the ancillary functions **b** and **c**. In each case the function defined on G restricts to the given function on each G_i .)

It is clear that $L(1_G) = 0$. Moreover, $g^{-1} = g_n^{-1} g_{n-1}^{-1} \cdots g_1^{-1}$, and

$$\begin{aligned} \alpha_g L(g^{-1}) &= \alpha_g \sum_{k=1}^n \alpha_{g_n^{-1} \cdots g_{n-k+2}^{-1}} L(g_{n-k+1}^{-1}) \\ &= \sum_{k=1}^n \alpha_{\overline{g_{n-k}}} \alpha_{g_{n-k+1}} L(g_{n-k+1}^{-1}) \\ &= \sum_{k=1}^n \alpha_{\overline{g_{n-k}}} L(g_{n-k+1}) \\ &= \sum_{k=1}^n \alpha_{\overline{g_{k-1}}} L(g_k) \\ &= L(g) \end{aligned}$$

Claim 7. Let $\gamma = \gamma_1 \cdot \gamma_2 \cdots \gamma_q$. Then $L(\gamma) = \sum_{k=1}^q \alpha_{\overline{\gamma_{k-1}}} L(\gamma_k)$. In particular if $L(\gamma) = 0$ then $L(\gamma_i) = 0$ for all i .

The claim is trivial if $q = 1$. If $q = 2$, and $\gamma_1 = g_1 \cdots g_p$ and $\gamma_2 = g_{p+1} \cdots g_r$, with g_i and g_{i+1} belonging to distinct free factors for each i , it is straightforward to show that both sides of the desired equation are equal to

$$\sum_{k=1}^r \alpha_{\overline{g_{k-1}}} L(g_k).$$

The claim follows by an easy induction on q .

Claim 8. Suppose that $g = g_1 \cdot g_2 \cdots g_n$ and $h = h_1 \cdot h_2 \cdots h_m$, and that $p \geq 0$ is the integer satisfying $g_i = h_i$ for $1 \leq i \leq p$, and either $g_{p+1} \neq h_{p+1}$ or $p = \min\{m, n\}$. Put $g_i = 1$ for $i > n$ and $h_i = 1$ for $i > m$. Then $\mathbf{c}(g, h) = L(\overline{g_p}) + \alpha_{\overline{g_p}} \mathbf{c}(g_{p+1}, h_{p+1})$.

If g_i and h_i belong to distinct free factors, $\mathbf{c}(g_i, h_i) = 0$.

Note first that if g_i and h_i belong to distinct free factors, we have

$$\begin{aligned}\alpha_{g_i} L(g_i^{-1} \cdot h_i) &= \alpha_{g_i} \left(L(g_i^{-1}) + \alpha_{g_i^{-1}} L(h_i) \right) \\ &= L(g_i) + L(h_i).\end{aligned}$$

Thus $\mathbf{c}(g_i, h_i) = 0$.

Write $g^{-1}h = g_n^{-1} \cdot g_{n-1}^{-1} \cdots g_{p+2}^{-1} \cdot (g_{p+1}^{-1} h_{p+1}) \cdot h_{p+2} \cdots h_m$.

Using Claim 34, it is then straightforward to show that

$$\alpha_g L(g^{-1}h) = \alpha_{\overline{g_{p+1}}} \left(L(g_{p+2} \cdots g_n) + L(g_{p+1}^{-1} h_{p+1}) + \alpha_{g_{p+1}^{-1} h_{p+1}} L(h_{p+2} \cdots h_m) \right).$$

Now

$$\begin{aligned}L(g) &= L(\overline{g_{p+1}}) + \alpha_{\overline{g_p} g_{p+1}} L(g_{p+2} \cdots g_n) \\ \text{and } L(h) &= L(\overline{h_{p+1}}) + \alpha_{\overline{h_p} h_{p+1}} L(h_{p+2} \cdots h_m).\end{aligned}$$

Combining these expressions, and recalling that $\overline{g_p} = \overline{h_p}$, we obtain

$$\begin{aligned}2\mathbf{c}(g, h) &= L(\overline{g_{p+1}}) + L(\overline{h_{p+1}}) - \alpha_{\overline{g_{p+1}}} L(g_{p+1}^{-1} h_{p+1}) \\ &= L(\overline{g_p}) + \alpha_{\overline{g_p}} L(g_{p+1}) + L(\overline{h_p}) + \alpha_{\overline{h_p}} L(h_{p+1}) - \alpha_{\overline{g_{p+1}}} L(g_{p+1}^{-1} h_{p+1}) \\ &= 2L(\overline{g_p}) + 2\alpha_{\overline{g_p}} \mathbf{c}(g_{p+1}, h_{p+1})\end{aligned}$$

which establishes the claim.

It is now clear that $\mathbf{c}(g, h) \in \Lambda$ for all $g, h \in G$.

We now verify the isosceles condition for $\mathbf{c}(g, h)$, $\mathbf{c}(g, k)$ and $\mathbf{c}(h, k)$. Keeping the notation for g and h as in Claim 35, write $k = k_1 \cdot k_2 \cdots k_l$. Let $q \geq 0$ be the integer for which $h_i = k_i$ for $0 \leq i \leq q$, and either $q = \min\{m, l\}$ or $h_{q+1} \neq k_{q+1}$. Assume without loss of generality that $p \leq q$. (This means that h and k share a common initial subword that is no shorter than that of g and h , when these elements of G are written as words in the free factors.) Observe that $g_i = k_i$ for $i \leq p$ and $g_{p+1} \neq k_{p+1}$, and therefore that $\overline{g_p} = \overline{h_p} = \overline{k_p}$.

Now Claim 35 gives

$$\begin{aligned}\mathbf{c}(g, h) &= L(\overline{g_p}) + \alpha_{\overline{g_p}} \mathbf{c}(g_{p+1}, h_{p+1}) \\ \mathbf{c}(g, k) &= L(\overline{g_p}) + \alpha_{\overline{g_p}} \mathbf{c}(g_{p+1}, k_{p+1}) \\ \text{and } \mathbf{c}(h, k) &= L(\overline{h_q}) + \alpha_{\overline{h_q}} \mathbf{c}(h_{q+1}, k_{q+1}).\end{aligned}$$

Suppose first that $\mathbf{c}(g, h) > \mathbf{c}(h, k)$. Then $L(\overline{g_p}) + \alpha_{\overline{g_p}} \mathbf{c}(g_{p+1}, h_{p+1}) > L(\overline{g_p}) + \alpha_{\overline{g_p}} L(h_{p+1} \cdots h_q) + \alpha_{\overline{h_q}} \mathbf{c}(h_{q+1}, k_{q+1})$. If $q = p$ this inequality reduces to $\mathbf{c}(g_{p+1}, h_{p+1}) > \mathbf{c}(h_{p+1}, k_{p+1})$, forcing $\mathbf{c}(h_{p+1}, k_{p+1}) = \mathbf{c}(g_{p+1}, h_{p+1})$ and thus $\mathbf{c}(h, k) = \mathbf{c}(g, k)$.

Otherwise $q \geq p + 1$, which implies $\mathbf{c}(g_{p+1}, h_{p+1}) > L(h_{p+1} \cdots h_q) \geq L(h_{p+1})$, contradicting Lemma 1.3(8).

Next suppose that $\mathbf{c}(g, k) > \mathbf{c}(g, h)$. Then $\mathbf{c}(g_{p+1}, k_{p+1}) > \mathbf{c}(g_{p+1}, h_{p+1})$. Therefore the left-hand side is non-zero, which forces g_{p+1} and k_{p+1} to belong to the same free factor, by Claim 35. If h_{p+1} belongs to a different free factor, then $h_{p+1} \neq k_{p+1}$ giving $q = p$. Thus $\mathbf{c}(h_{p+1}, k_{p+1}) = 0 = \mathbf{c}(g_{p+1}, h_{p+1})$, which implies $\mathbf{c}(h, k) = \mathbf{c}(g, h)$.

If h_{p+1} belongs to the same free factor as g_{p+1} and k_{p+1} then $\mathbf{c}(h_{p+1}, k_{p+1}) = \mathbf{c}(g_{p+1}, h_{p+1}) < \mathbf{c}(g_{p+1}, k_{p+1})$. Again we must have $q = p$, for otherwise $q \geq p + 1$, so that $g_{p+1} \neq k_{p+1} = h_{p+1}$ and thus $L(k_{p+1}) \geq \mathbf{c}(g_{p+1}, k_{p+1}) > \mathbf{c}(h_{p+1}, k_{p+1}) = L(k_{p+1})$ by Lemma 1.3(5), a contradiction. The equality $\mathbf{c}(h_{p+1}, k_{p+1}) = \mathbf{c}(g_{p+1}, h_{p+1})$ now gives the required $\mathbf{c}(h, k) = \mathbf{c}(g, h)$.

Finally, suppose that $\mathbf{c}(h, k) > \mathbf{c}(g, h)$. If $q \geq p + 1$ then $h_{p+1} = k_{p+1}$ giving $\mathbf{c}(g, h) = \mathbf{c}(g, k)$. Otherwise $q = p$ and this inequality yields $L(\overline{h_p}) + \alpha_{\overline{h_p}} \mathbf{c}(h_{p+1}, k_{p+1}) > L(\overline{g_p}) + \alpha_{\overline{g_p}} \mathbf{c}(g_{p+1}, h_{p+1})$, giving $\mathbf{c}(h_{p+1}, k_{p+1}) > \mathbf{c}(g_{p+1}, h_{p+1}) = \mathbf{c}(g_{p+1}, k_{p+1})$. This implies $\mathbf{c}(g, h) = \mathbf{c}(g, k)$.

The remaining cases, such as $\mathbf{c}(g, k) > \mathbf{c}(h, k)$, follow from those we have considered by swapping the roles of h and k .

This completes the proof that L is an α -affine length function.

We now prove that if the given actions of G_i are free and without inversions, then so is that of G . This follows easily from our final claim:

Claim 9. *Suppose that $r > 1$, and $g = g_1 \cdots g_r$ is cyclically reduced (i.e. g_1 and g_r belong to distinct free factors). If $L(g_i) \neq 0$ for all i then g is hyperbolic and rigid.*

We have $L(g) \neq 0$. For otherwise, by Claim 34, if $g = g_1 \cdots g_r$ then $L(g_i) = 0$ for all i , contradicting the non-triviality of g . Let x be the basepoint of the Λ -tree X corresponding to the length function L on G . Observe that if $g^n x = g^m x$ for some $m < n$ then $L(g^k) = 0$ for some $k \geq 1$, giving $0 = L(g^k) = L(g) + \alpha_g L(g) + \cdots + \alpha_{g^{n-1}} L(g) \geq L(g)$ — here we have used the assumption that g is cyclically reduced and has length at least 2. Thus $L(g) = 0$, which is impossible. Note also that the equality

$$L(g^2) = L(g \cdot g) = L(g) + \alpha_g L(g),$$

which follows from Claim 1, implies $[x, gx, g^2x]$, giving $x \in A_g$. Since $A_g \neq \emptyset$, g cannot be a nesting reflection or an inversion. This means that g restricted to the invariant subtree $\cup_{n \in \mathbb{Z}} [g^{n-1}x, g^n x] \subseteq A_g$ is hyperbolic. To establish hyperbolicity of g as an automorphism of X , it suffices to show this set in fact coincides with A_g .

So suppose otherwise. Observe that X is spanned by points of the form γx where γ ranges through G . Replacing g by its inverse if necessary, our supposition gives, for some $\gamma \in G$,

$$[x, g^n x, \gamma x] \quad \forall n \geq 1.$$

Now

$$\begin{aligned} [x, \gamma_1 x, \gamma_2 x] = [x, \gamma_2 x] &\Leftrightarrow L(\gamma_1) + \alpha_{\gamma_1} L(\gamma_1^{-1} \gamma_2) = L(\gamma_2) \\ &\Leftrightarrow L(\gamma_1) = \mathbf{c}(\gamma_1, \gamma_2) \end{aligned}$$

for $\gamma_1, \gamma_2 \in G$, so replacing γ_1 by g^n and γ_2 by γ we get

$$\mathbf{c}(g^n, \gamma) = L(g^n) \quad \forall n \geq 1. \tag{1}$$

To show that g is hyperbolic (with respect to the action on X), it suffices to show that g^n is hyperbolic for some n , since g is hyperbolic with respect to the action on $\cup_{n \in \mathbb{Z}} [g^{n-1}x, g^n x]$, and thus cannot act as an inversion or a nesting reflection. So if g is an initial subword of γ (i.e. if γ can be written in the form $g \cdot g^{-1}\gamma$), then we can replace g by a sufficiently large power of g to ensure that this is not the case. Now note that the longest common initial subword $\overline{g_p}$ of g and γ is the same as that g^n and γ . So by Claim 35, we have

$$\mathbf{c}(g^n, \gamma) = L(\overline{g_p}) + \alpha_{\overline{g_p}} \mathbf{c}(g_{p+1}, \gamma_{p+1}) = \mathbf{c}(g, \gamma). \tag{2}$$

But (1) and (2) give $L(g^n) = L(g)$ for $n \geq 1$ — this is absurd since in particular $L(g) + \alpha_g L(g) = L(g \cdot g) = L(g^2) = L(g)$ implies $L(g) = 0$, which is impossible.

Since conjugates of hyperbolic automorphisms are hyperbolic, parts (1) and (2) are now clear.

Now suppose that L_i is a regular length function for each i , and that $g, h \in G$. Then with the notation of Claim 35, if g_{p+1} and h_{p+1} belong to distinct free factors, we have $\mathbf{c}(g, h) = L(\overline{g_p})$, $g = \overline{g_p} \cdot (g_{p+1} \cdots g_n)$ and $h = \overline{g_p} \cdot (h_{p+1} \cdots h_m)$, and thus $L(g) = L(\overline{g_p}) + \alpha_{\overline{g_p}} L(g_{p+1} \cdots g_n)$ and $L(h) = L(\overline{g_p}) + \alpha_{\overline{g_p}} L(h_{p+1} \cdots h_m)$. Otherwise, let u_{p+1} be the element of the same free factor as g_{p+1} and h_{p+1} satisfying $g_{p+1} = u_{p+1} g'_{p+1}$, $h_{p+1} = u_{p+1} h'_{p+1}$, $L(g_{p+1}) = L(u_{p+1}) + \alpha_{u_{p+1}} L(g'_{p+1})$, $L(h_{p+1}) = L(u_{p+1}) + \alpha_{u_{p+1}} L(h'_{p+1})$ and $L(u_{p+1}) = \mathbf{c}(g_{p+1}, h_{p+1})$. It is routine to verify that $\mathbf{c}(g, h) = L(u)$ where $u = \overline{g_p} \cdot u_{p+1}$, and that $L(g) = L(u) + \alpha_u L(g')$, $L(h) = L(u) + \alpha_u L(h')$, where $g = ug'$ and $h = uh'$. This shows that L is regular. \square

The argument given shows that if the actions of G_i are both free and rigid then so is that of G . In fact if the actions of G_i are merely assumed to be rigid and if for each i there exists $x_i \in X$ which is not fixed by any $g_i \in G_i$, then taking these points as the basepoints for the length functions L_i , it follows that all cyclically reduced $g \in G$ of length at least 2 (as a word in the free product) are hyperbolic and rigid. Thus G acts rigidly.

Corollary 3.2. (i). Let G_i be an $\text{ATF}(\mathbb{R})$ group for $i \in I$. Then $*_{i \in I} G_i$ is an $\text{ATF}(\mathbb{R})$ group.

(ii). The class of ATF groups is closed under free products.

Proof: Part (1) is immediate from the theorem. For part (2) it suffices to show that given ordered abelian groups Λ_i ($i \in I$) there exist Λ and embeddings $h_i : \Lambda_i \rightarrow \Lambda$ such that every automorphism of Λ_i extends to one of Λ , and such that if free $\alpha^{(i)}$ actions of G_i on Λ_i -trees are given, the induced actions of G_i on $\Lambda \otimes_{\Lambda_i} X_i$ are also free.

Endow I with a linear order. Let Λ be the subgroup of $\prod_{i \in I} \Lambda_i$ consisting of those (λ_i) with well-ordered support. This makes Λ an ordered abelian group. Moreover each Λ_i embeds in Λ via $\lambda_i \mapsto (\delta_{ij} \lambda_i)_{j \in I}$ (where δ_{ij} is the Kronecker delta function) and automorphisms α_* of Λ_i extend to automorphisms $\bar{\alpha}_*$ in an obvious way. By Theorem 1.5, there is an induced action of G_i on a Λ -tree \bar{X}_i . Moreover, choosing a basepoint $x_i \in X_i$, and letting $L_i = L_{x_i}$ and $\bar{L}_i = L_{\phi_i(x_i)}$ (where ϕ_i denotes the embedding of X_i in \bar{X}_i), we have $\bar{L}_i = h_i L_i$, by Theorem 1.5(3)(c). The ancillary functions with respect to the actions on X_i and \bar{X}_i are similarly related, and will be similarly notated.

By Proposition 1.15, $\overline{\text{Rad}}^r(g)$ is spanned by terms that are \mathbb{Z} -linear combinations of $\bar{L}_i = h_i L$ and $\bar{\mathbf{a}}_i = h_i \mathbf{a}_i$. It follows that $\overline{\text{Rad}}^r(g)$ is spanned by $h_i(\text{Rad}^r(g))$.

If $\bar{\lambda} = (\lambda_j)_{j \in I} \in \Lambda$ satisfies $(1 - \bar{\alpha}_g)(\bar{\lambda}) = \bar{\mathbf{b}}_i(g)$ then $\lambda_i - \alpha_g^{(i)}(\lambda_i) = \mathbf{b}_i(g)$. If further $\bar{\lambda} \in \overline{\text{Rad}}^r(g)$, then $\lambda_i \leq L_i(\gamma) - \mathbf{a}_i(g) - \alpha_\gamma \mathbf{a}_i(\gamma^{-1}g\gamma)$ for some γ . It follows that $\lambda_i \in \text{Rad}^r(g)$, so that g has a fixed point by Proposition 1.14, a contradiction. Therefore $(1 - \bar{\alpha}_g^{(i)})^{-1}(\bar{\mathbf{b}}_i(g)) \cap \overline{\text{Rad}}^r(g) = \emptyset$, so that hyperbolicity is preserved by the base change functor in this case. By Corollary 1.13(1), the action of G_i on \bar{X}_i is free.

The result now follows from Theorem 3.1. \square

3.2 Ultraproducts

Theorem 3.3. Let G_i be a group with an $\alpha^{(i)}$ -affine action on a Λ_i -tree (X_i, d_i) and let \mathcal{D} be an ultrafilter in I . Then $G = \prod_{i \in I} G_i / \mathcal{D}$ has an induced α -affine action on a Λ -tree $(\Lambda = \prod_{i \in I} \Lambda_i / \mathcal{D})$.

If, for almost all i , the given actions of G_i are respectively

(i). free (and without inversions),

(ii). rigid,

(iii). or regular

then so is that of G .

Proof: Much of the proof is a routine use of ultraproducts, and we will omit most of the details. (See [10, §5.5] or [3, §3] for the necessary background.)

Define a Λ -tree (X, d) as follows. The set X is the ultraproduct of the X_i , so X consists of $\langle x_i \rangle$ where $x_i \in X_i$ for all i , and $d(\langle x_i \rangle, \langle y_i \rangle) = \langle d_i(x_i, y_i) \rangle$. (Here we denote by $\langle x_i \rangle$ the equivalence class of the I -sequence $(x_i)_{i \in I}$ where two I -sequences are equivalent if their agreement set is an element of \mathcal{D} ; similar notational conventions apply to the other ultra-objects we consider.)

Using the criterion

$$y \in [x, z] \Leftrightarrow d(x, z) = d(x, y) + d(y, z),$$

it is easy to verify that

$$[\langle x_i \rangle, \langle z_i \rangle] = \langle [x_i, z_i] \rangle. \quad (3)$$

If ζ_i is an isometric isomorphism $[0, \lambda_i] \rightarrow [x_i, z_i]$, then $\zeta : [0, \lambda] \rightarrow [\langle x_i \rangle, \langle z_i \rangle]$ defined by

$$\zeta \langle t_i \rangle = \langle \zeta_i(t_i) \rangle$$

is an isometric isomorphism (where $\lambda = \langle \lambda_i \rangle$). Therefore (X, d) is geodesic. It is similarly straightforward to check that for $v \in X$, we have $(x \cdot y)_v \in \Lambda$, and that X is 0-hyperbolic.

Putting

$$\alpha_{\langle g_i \rangle} \langle \lambda_i \rangle = \langle \alpha_{g_i}^{(i)}(\lambda_i) \rangle,$$

we obtain a homomorphism $G \rightarrow \text{Aut}^+(\Lambda)$ and there is an induced α -affine action of G on X given by

$$\langle g_i \rangle \cdot \langle x_i \rangle = \langle g_i x_i \rangle.$$

If the given actions of G_i are free, it is easy to see that the action of G is. Using the description of segments in X in (3) one can show that rigidity is likewise preserved by the ultraproduct action. To see that regularity is preserved by the action of G , choose a basepoint $v_i \in X_i$, put $L_i(g_i) = d_i(v_i, g_i v_i)$, and suppose that for $g_i, h_i \in G_i$ there exists $u_i \in G_i$ such that $\mathbf{c}(g_i, h_i) = L_i(u_i)$. Suppose further that $L_i(g_i) = L_i(u_i) + \alpha_{u_i} L_i(g'_i)$ and $L_i(h_i) = L_i(u_i) + \alpha_{u_i} L_i(h'_i)$ for some g'_i and h'_i such that $g_i = u_i g'_i$ and $h_i = u_i h'_i$. Then one has $\mathbf{c}(\langle g_i \rangle, \langle h_i \rangle) = L(\langle u_i \rangle)$, where $L = L_{\langle v_i \rangle}$, $L\langle g_i \rangle = L\langle u_i \rangle + \alpha_{\langle u_i \rangle} L\langle g'_i \rangle$, $L\langle h_i \rangle = L\langle u_i \rangle + \alpha_{\langle u_i \rangle} L\langle h'_i \rangle$, and $\langle u_i \rangle \langle g'_i \rangle = \langle g_i \rangle$ and $\langle u_i \rangle \langle h'_i \rangle = \langle h_i \rangle$. \square

Theorem 3.4. (i). A group is locally in ATF if and only if it is in ATF.

(ii). A group is fully residually in ATF if and only if it is in ATF.

Proof: The necessity in both parts is clear, so we will focus on the sufficiency.

(1) Let G_i ($i \in I$) denote the finitely generated subgroups of G , and assume that each G_i is in ATF. By Theorem 3.3, the group ${}^*G = \prod_{i \in I} G_i / \mathcal{D}$ is in ATF, so it suffices to show that \mathcal{D} can be chosen such that G embeds in *G . Write $a_x = \{i \in I : x \in G_i\}$ for $x \in G$, and note that since $i_0 \in \bigcap_{i=1}^n a_{x_i}$ where $G_{i_0} = \langle x_1, \dots, x_n \rangle$, the sets a_x ($x \in G$) have the finite intersection property. Using Zorn's Lemma, we can find a maximal ultrafilter \mathcal{D} that contains each a_x . Now given $x \in G$, and putting $x_i = x$ for all i for which $x_i \in G_i$ (and $x_i = 1$ otherwise), we see that $x \mapsto (x_i)_{i \in I} / \mathcal{D}$ is a well-defined monomorphism.

(2) Consider now the set of normal subgroups N_i ($i \in I$) of G for which G/N_i is in ATF, and suppose that for all finite subsets X of $G \setminus 1$ we have $N_i \cap X = \emptyset$ for some i . Again we propose to embed G in a suitable ultraproduct of ATF groups, from which the required result can be deduced from Theorem 3.3

For $x \in G \setminus 1$ put $a_x = \{i \in I : x \notin N_i\}$. As in part (1), the sets a_x are easily seen to have the finite intersection property, whence an ultrafilter \mathcal{D} containing each a_x . For $x \in G$ we put $x_i = x$ for all i and map $x \mapsto (x_i N_i)_{i \in I} / \mathcal{D}$; this gives an embedding of G in $\prod_{i \in I} (G/N_i) / \mathcal{D}$. \square

3.3 Extending an isometric subaction to an affine action

The following theorem and proof follow the idea of [20, Corollaire 2(2)]

Recall that an isometric action of G is *abelian* if $\ell(gh) \leq \ell(g) + \ell(h)$ for all $g, h \in G$.

Theorem 3.5. Let G be a group, and N a normal subgroup. Suppose that N has a minimal non-abelian isometric action on a Λ -tree X . Let $\alpha : G \rightarrow \text{Aut}^+(\Lambda)$ be a homomorphism, with $N \leq \ker \alpha$. Then there is an α -affine action of G on X extending the original action of N if and only if, for $g \in N$ and $\gamma \in G$, we have

$$\ell(\gamma g \gamma^{-1}) = \alpha_\gamma \ell(g). \quad (4)$$

Proof: The sufficiency is straightforward to show.

Conversely, let ℓ denote the hyperbolic length function arising from the isometric action of N . Fix $\gamma \in G$ and consider the Λ -tree (X, d_1) where $d_1 = \alpha_{\gamma^{-1}} \circ d$ and the $(d_1$ -isometric) action of N given by $g \cdot x = \gamma g \gamma^{-1} x$. Then $d_1(x, g \cdot x) = \alpha_{\gamma^{-1}} d(x, \gamma g \gamma^{-1} x)$. It follows, using (4) denoting the associated hyperbolic length function by ℓ_1 , that $\ell_1(g) = \ell(g)$, whence a unique N -equivariant isometry $\phi = \phi_\gamma : (X, d) \rightarrow (X, d_1)$, by [10, Theorem 3.4.1]. Thus $d(\phi(x), \phi(y)) = \alpha_\gamma d(x, y)$, which means that ϕ is an affine automorphism of (X, d) with dilation factor α_γ .

Now consider $\gamma_1, \gamma_2 \in G$. Since ϕ_{γ_1} is an N -equivariant isometry $(X, d) \rightarrow (X, \alpha_{\gamma_1^{-1}} d)$, it is also an N -equivariant isometry $(X, \alpha_{\gamma_2^{-1}} d) \rightarrow (X, \alpha_{\gamma_2^{-1}} \alpha_{\gamma_1^{-1}} d)$. Therefore $\phi_{\gamma_1} \phi_{\gamma_2}$ is an N -equivariant isometry $(X, d) \rightarrow (X, \alpha_{\gamma_2^{-1} \gamma_1^{-1}} d)$. Since such an isometry is unique, we must have $\phi_{\gamma_1 \gamma_2} = \phi_{\gamma_1} \phi_{\gamma_2}$.

We therefore have an affine action of G on X by putting $\gamma x = \phi_\gamma(x)$. Moreover if $\gamma \in N$, then $x \mapsto \gamma x$ is an N -equivariant isometry $(X, d) \rightarrow (X, d)$ (since α_γ is trivial in this case), whence $\phi_\gamma = \gamma$. This means that the action of G extends the action of N . \square

We remark that Chiswell has recently shown that $\text{ITF}(\mathbb{R}^n)$ groups are right orderable [9]. Since $\text{Aut}^+(\Lambda)$ is right orderable, it follows (via the map α) that $\text{ATF}(\Lambda)$ groups are $\text{ITF}(\Lambda)$ -by-(right orderable); hence $\text{ATF}(\mathbb{R}^n)$ groups are right orderable.

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